Notes on Complex Integration
Stein and Shakarchi Chapter 1 Section 3

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In complex analysis, **integration** means *integration along curves*. In our presentation (and that of Stein and Shakarchi) the basis of complex integration does not really use anything beyond basic Riemann integration of real functions from elementary calculus. I felt, however, that there might be some value in a brief review of real integration and the addition of some details glossed over by Stein and Shakarchi.

1 Real Integration

Given a real valued function $f : \Gamma \to \mathbb{R}$ defined on a curve $\Gamma \subset \mathbb{R}^2$, we may attempt (and succeed in many instances) to define integration of f on Γ by

$$\int_\Gamma f = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{k} f(p_j^*)\, \mathcal{H}^1(\Gamma_j). \tag{1}$$

In this expression $\mathcal{P} = \{\Gamma_j\}_{j=1}^k$ is a partition of the curve Γ so that

$$\Gamma = \bigcup_{j=1}^{k} \Gamma_j \qquad \text{and} \qquad \mathcal{H}^1(\Gamma_i \cap \Gamma_j) = 0 \quad \text{when} \quad i \neq j;$$

$\|\mathcal{P}\| = \max\{\operatorname{diam}\Gamma_j : j = 1, 2, \ldots, k\}$, the point $p_j^* \in \Gamma_j$ is an evaluation point, and $\mathcal{H}^1$ is length measure in $\mathbb{R}^2$, i.e., one-dimensional Hausdorff measure. The expression on the right of (1) is called a **Riemann sum** (of course).

It is generally a good idea to keep this kind of definition of integration (in terms of Riemann sums) in the back of one's mind, and it will be the background for all (or at least most) integration we consider. Moreover, when we integrate on curves in complex analysis, we will borrow some of the notation from (1), though the complex integration on curves we consider will be fundamentally different in several ways.

Recall that typical assumptions concerning the curve Γ and the function $f : \Gamma \to \mathbb{R}$ are that Γ admits a parameterization $\alpha : [a, b] \to \Gamma$ with $\alpha \in C^1([a, b] \to \mathbb{R}^2)$ and $f \in C^0(\Gamma)$. It is also common to consider Γ to be piecewise C^1, that is to say a concatenation of curves admitting C^1 parameterizations or, more precisely, that there are partitions $\mathcal{Q} = \{\Gamma_j\}_{j=1}^k$ of Γ and

$$a = t_0 < t_1 < \cdot < t_k = b$$

of $[a, b]$ for which there exist parameterizations $\alpha_j : [t_{j-1}, t_j] \to \Gamma_j$ with $\alpha_j \in C^1([t_{j-1}, t_j] \to \mathbb{R}^2)$ for $j = 1, 2, \ldots, k$. In these cases we can express the integral in (1) using the **change of variables formula**

$$\int_\Gamma f = \int_a^b f \circ \alpha(t)\ |\alpha'(t)|\, dt = \sum_{j=1}^k \int_{t_{j-1}}^{t_j} f \circ \alpha_j(t)\ |\alpha_j'(t)|\, dt$$

or simply

$$\int_\Gamma f = \int_{(a,b)} f \circ \alpha\ |\alpha'|. \tag{2}$$

The parameterization α may be considered to **orient** the curve Γ, but the resulting orientation has nothing to do with the value of the integral in (1); this is also reflected in the appearance of the Euclidean norm in the change of variables formula for the integral.

The notation in (2) may be interpreted to emphasize the consideration of the interval $(a, b) \subset \mathbb{R}$ as a geometric object without orientation. Of course there is always a natural orientation in $\mathbb{R}$ according to which we write, for example,

$$\int_a^b g(\xi)\, d\xi = -\int_b^a g(\xi)\, d\xi. \tag{3}$$

For $a < b$, integrals like the one on the right in (3) are not naturally expressed in the notation appearing in (2).

If $\alpha : [a, b] \to \Gamma$ parameterizes a curve Γ, we can almost always “trade in” α for an **arclength parameterization** $\gamma : [0, L] \to \Gamma$ where

$$L = \int_a^b |\alpha'(t)|\, dt = \mathcal{H}^1(\Gamma) \quad \text{is the length of } \Gamma. \tag{4}$$

More precisely $\gamma = \gamma(s) = \alpha(t)$ with

$$s = \int_a^t |\alpha'(\tau)|\, d\tau.$$

Like the correspondence $z = x + iy$, $f = u + iv$, we will use the notation associated with "trading in" $\alpha : [a, b] \to \Gamma$ for an arclength parameterization $\gamma : [0, L] \to \Gamma$ in a more or less standardized fashion.

Hopefully the above summarizes some aspects of real integration that will be helpful to keep in mind as they are applied to the treatment of complex integration.

2 Real Integrals of Complex Valued Functions

At the risk of creating confusion, I am going to introduce a kind of "hybrid integral" that is a small generalization of the real integrals from calculus. When we talk about **complex integration**, these are **not** the integrals we have in mind. Nevertheless, we will use these hybrid integrals.

Given $g : [a, b] \to \mathbb{C}$ with $g = h + ik \in C^0([a, b] \to \mathbb{C})$ we set

$$\int_a^b g(t)\, dt = \int_a^b h(t)\, dt + i \int_a^b k(t)\, dt.$$

These integrals inherit many (obvious) properties from one-dimensional real integrals. I will not try to list these properties now, but I will try to note them when they are used/needed.

3 Complex Integration

For this section, let α (or as Stein writes z) parameterize a curve in $\mathbb{C}$. That is, $\alpha : [a, b] \to \Gamma \subset \mathbb{C}$.

Generally we will assume $\alpha : [a, b] \to \Gamma$ is a **regular C^1 curve**. This means

(i) $x, y \in C^1[a, b]$ where $x = \operatorname{Re} \alpha$ and $y = \operatorname{Im} \alpha$, and

(ii) $\alpha' = x' + iy' \neq 0$.

Condition (ii) is what it means for the curve to be **regular**. Note that the derivative α' is not a complex derivative but rather a "real derivative of a complex valued function."

Exercise 1 *If α is a regular parameterization of a C^1 curve, then for each $t \in (a,b)$, there is a complex number $L \in \mathbb{C}$ for which*

$$\lim_{h\to 0} \frac{\alpha(t+h)-\alpha(t)}{h} = L.$$

Note that $h \in \mathbb{R}$ in this limit. Moreover, $L = x'(t) + iy'(t)$.

Piecewise regular C^1 curves are also important/useful. In this case we have a partition $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a,b]$ with $\alpha : [a,b] \to \Gamma$ and each restriction

$$\alpha_j = \alpha\Big|_{[t_{j-1},t_j]} \to \Gamma \qquad \text{for} \qquad j = 1, 2 \ldots, k$$

is a regular C^1 parameterization.

A **one-dimensional change of variables** is a continuously differentiable bijection $\xi : [a,b] \to [c,d]$. Notice ξ is either increasing or decreasing.

Exercise 2 *Given a regular C^1 curve parameterized by*

$$\alpha : [a,b] \to \Gamma \qquad \text{and} \qquad \tilde{\alpha} : [c,d] \to \Gamma,$$

let us say $\alpha \sim \tilde{\alpha}$ if there exists a one-dimensional change of variables $\xi : [a,b] \to [c,d]$ for which $\alpha = \tilde{\alpha} \circ \xi$. Show that " $\sim$ " is an equivalence relation.

Complex integrals **always** depend on orientation: Given a piecewise regular C^1 curve $\Gamma \subset \mathbb{C}$ **with orientation** determined by an arclength parameterization $\gamma : [0,L] \to \Gamma$ and a function $f \in C^0(\Gamma)$, we define **the** (complex) **integral of** f **over** Γ by

$$\int_\gamma f = \int_0^L f \circ \gamma(s)\ \gamma'(s)\, ds. \tag{5}$$

Note that even though $\gamma'(s)$ is a unit vector the integrand

$$f \circ \gamma(s)\ \gamma'(s)$$

is a complex number depending on the orientation of Γ. In particular, the integral in definition (5) is a "hybrid" real integral of a complex valued function. Recall also the equivalence of the absolute value in $\mathbb{C}$ with the Euclidean norm of the corresponding point in $\mathbb{R}^2$ according to which

$$|\alpha'| = |(x', y')| \qquad \text{and} \qquad L = \int_{(a,b)} |\alpha'|$$

is the **length** of a curve $\Gamma \subset \mathbb{C}$.

Lemma 1 *If* $\alpha = \gamma \circ \xi$ *for a one-dimensional change of variables* ξ *with* $\xi' > 0$, *then*

$$\int_0^L f \circ \gamma(s)\ \gamma'(s)\, ds = \int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt.$$

That is, the hybrid integral

$$I[\alpha] = \int_a^b f \circ \alpha(t)\, \alpha'(t)\, dt$$

is constant on the equivalence class of parameterizations differing by an orientation preserving change of variables.

Proof:

$$\begin{aligned} \int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt &= \int_a^b f \circ \gamma \circ \xi(t)\ (\gamma \circ \xi)'(t)\, dt \\ &= \int_a^b f \circ \gamma \circ \xi(t)\ \gamma' \circ \xi(t)\ \xi'(t)\, dt \\ &= \int_0^L f \circ \gamma(s)\ \gamma'(s)\, ds. \qquad \Box \end{aligned}$$

In this proof we have used two minor generalizations of familiar assertions.

Exercise 3 *Verify the following concerning change of variables: If*

(i) $\alpha : [a, b] \to \Gamma$ *parameterizes a curve* $\Gamma \subset \mathbb{C}$,

(ii) $\beta : [c, d] \to \Gamma$ *parameterizes the same curve,*

(iii) $\xi : [a, b] \to [c, d]$ *is a change of variables, and*

(iv) $g : \Gamma \to \mathbb{C}$ *is continuous,*

then

(a) *The usual chain rule holds for the composition of a complex valued function of a real variable and a real valued function of a real variable:*

$$(\beta \circ \xi)' = (\beta' \circ \xi)\ \xi'.$$

(b) *The usual change of varibles formula holds for the (hybrid) integral of a complex valued function on a complex curve subject to a real change of variable:*

$$\int_a^b g(\beta \circ \xi(t))\ \xi'(t)\,dt = \int_c^d g \circ \beta(\xi)\,d\xi.$$

In view of Lemma 1 we also write

$$\int_\alpha f = \int_a^b f \circ \alpha(t)\ \alpha'(t)\,dt$$

for any $\alpha : [a, b] \to \Gamma$. (The integral depends on the orientation determined by α but not on the parameterization α in any other way.)

Proposition 1 (Proposition 3.1 in S&S) *The following hold for complex integrals:*

(i) *(linearity)*

$$\int_\alpha (af + bg) = a \int_\alpha f + b \int_\alpha g.$$

(ii) *(reverse orientation)*

$$\int_{-\alpha} f = -\int_\alpha f.$$

Note: The parameterization $-\alpha$ here can be taken as $-\alpha : [0, 1] \to \gamma$ by $-\alpha(t) = \alpha((1 - t)b + ta)$. A reverse parameterization, i.e., a reverse of orientation in a complex integral is denoted by α^- by Stein.

(iii) *(basic inequality for integral estimates) If $|f(z)| \le M$ for $z \in \Gamma$, then*

$$\left| \int_\alpha f \right| \le M\ \text{length}(\Gamma).$$

Here is a (rather more detailed version of the) proof of the basic inequality (iii) given by Stein:

$$\begin{aligned}
\left|\int_\alpha f\right| &= \left|\int_a^b f\circ\alpha(t)\ \alpha'(t)\,dt\right| \\
&= \left|\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^k f\circ\alpha(t_j^*)\ \alpha'(t_j^*)\,\mathcal{H}^2(\Gamma_j)\right| \\
&= \lim_{\|\mathcal{P}\|\to 0}\left|\sum_{j=1}^k f\circ\alpha(t_j^*)\ \alpha'(t_j^*)\,\mathcal{H}^2(\Gamma_j)\right| \\
&\le \lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^k \left|f\circ\alpha(t_j^*)\ \alpha'(t_j^*)\right|\ \mathcal{H}^2(\Gamma_j) \\
&= \lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^k \left|f\circ\alpha(t_j^*)\right|\ \left|\alpha'(t_j^*)\right|\ \mathcal{H}^2(\Gamma_j) \\
&\le N\lim_{\|\mathcal{P}\|\to 0}\sum_{j=1}^k \left|\alpha'(t_j^*)\right|\ \mathcal{H}^2(\Gamma_j) \\
&= M\int_a^b |\alpha'(t)|\,dt \\
&= M\,\text{length}(\Gamma). \qquad \square
\end{aligned}$$

Theorem 1 (Theorem 3.2 in S&S, integration of a complex derivative) *If* $f : \Omega \to \mathbb{C}$ *is differentiable and* $\Gamma \subset \Omega$ *with orientation* α, *then*

$$\int_\alpha f' = f\circ\alpha(b) - f\circ\alpha(a).$$

Proof: Let us first note that the fundamental theorem of calculus holds for hybrid integrals (i.e., real integrals of complex valued functions) in the form

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

We will need to use this. By definition

$$\int_\alpha f' = \int_a^b f'\circ\alpha(t)\ \alpha'(t)\,dt.$$

As we look at the integrand on the right, we see the product of a complex derivative and a real derivative. Naturally, we expect there should be a chain rule

$$(f \circ \alpha)' = (f' \circ \alpha)\ \alpha'$$

in this case, and there is, but technically we haven't proved it yet. See the exercise after we finish this proof. Assuming this chain rule, we can apply the fundamental theorem of calculus mentioned at the beginning of this proof:

$$\begin{aligned}\int_\alpha f' &= \int_a^b f' \circ \alpha(t)\ \alpha'(t)\, dt \\ &= \int_a^b (f \circ \alpha)'(t)\, dt \\ &= f \circ \alpha\Big|_a^b \\ &= f \circ \alpha(b) - f \circ \alpha(a). \qquad \square\end{aligned}$$

For a piecewise C^1 curve Γ, (you can) apply this result on the partitition pieces and get a telescoping sum.

Exercise 4 *Here are precise statements of simple results used in the foregoing proof.*

(a) *If $g : [a, b] \to \mathbb{C}$ has continuous (real) derivative $g' = h' + ik'$, then*

$$\int_a^b g'(t)\, dt = g(b) - g(a).$$

This is a version of the fundamental theorem of calculus for complex valued functions $g \in C^1([a, b] \to \mathbb{C})$.

(b) *(another chain rule) If $f : \Omega \to \mathbb{C}$ is differentiable and $\alpha : [a, b] \to \Omega$ has continuous (real) derivative $\alpha' = x' + iy'$, then*

$$\frac{d}{dt}(f \circ \alpha) = (f' \circ \alpha)\ \frac{d}{dt}\alpha.$$

Corollary 2 (Corollary 3.3 in S&S) *If Γ is a* **closed curve** *with $\Gamma \subset \Omega$ and $f : \Omega \to \mathbb{C}$ is differentiable, then*

$$\int_\alpha f' = 0. \tag{6}$$

Proof: **Closed** means $\alpha(b) = \alpha(a)$. $\Box$

This says the derivative of a complex differentiable function is closely related to a conservative vector field in calculus.

Corollary 3 *There does not exist a differentiable function* $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ *with* $f'(z) = 1/z$.

Proof: Assume by way of contradiction that such a function f, a primitive of $1/z$, does exist. Then we can consider the unit circle parameterized by $\gamma(t) = e^{it}$. Since $\gamma'(t) = ie^{it}$, we have

$$\int_\gamma f' = \int_\gamma \frac{1}{z} = \int_0^{2\pi} \frac{1}{e^{it}}\; i\; e^{it}\, dt = 2\pi i \neq 0. \qquad \Box$$

Stein says that (6) is a manifestation of Cauchy's theorem which says, in certain cases, for a differentiable function $f : \Omega \to \mathbb{C}$ one has

$$\int_\alpha f = 0.$$

I suppose this is an okay suggestion, as far as it goes. You can see, however, that the function $g(z) = 1/z$ on the punctured plane gives some kind of "counterexample." The point, it turns out, is that there is a singularity, a pole, that the curve we chose goes around.

Exercise 5 *Let* Γ *be the boundary of the square* $U = \{z = x + iy : 1 < x, y < 2\}$. *Compute*

$$\int_\gamma \frac{1}{z}.$$

Theorem 2 *(Corollary 3.4 of S&S) If* $f : \Omega \to \mathbb{C}$ *is differentiable and* Ω *is connected with* $f' \equiv 0$, *then* f *is constant.*

Proof: Recall that connected open subsets of $\mathbb{C}$ are also path connected. We claim that given two points z_0 and z in a connected open domain Ω, there exists a smooth regular path $\Gamma \subset \Omega$ connecting z_0 to z. If this is correct, then

$$\begin{aligned} f(z) - f(z_0) &= f \circ \alpha(b) - f \circ \alpha(a) \\ &= \int_\alpha f' \\ &= 0. \end{aligned}$$

So $f(z) \equiv f(z_0)$. $\Box$