The Geometric Series and Boundary Behavior of Complex Power Series

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We have established the basic convergence/divergence theorem for complex power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

involving the Hadamard radius R. This result defines the **region of absolute convergence** $D_R(0)$ on which

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines a holomorphic function. This result also defines a **region of divergence** $\mathbb{C}\backslash\overline{D_R(0)}$ on which the series in (1) diverges. We have observed that this divergence can take two distinct forms. One case is that in which the values of the partial sums

$$S_N = \sum_{n=0}^{N} a_n z^n$$

actually converge to ∞ (in the Riemann sphere). The complementary case is when some subsequence of the partial sums remains bounded. For a given series (1) we have defined the region

$$\Omega_S = \{z \in \mathbb{C}\backslash\overline{D_R(0)} : S_N \to \infty\}$$

and called it the Savio region after Daniel Savio who was the first one (in our complex analysis class of Spring 2022) to ask about the nature of the divergence. We have no proof that $\Omega_S \neq \phi$ in general, but we do know $\Omega_S = \mathbb{C}\backslash\overline{D_1(0)}$ for the geometric series

$$\sum_{n=0}^{\infty} z^n.$$

This is because for $|z| > 1$, we have for the geometric series

$$S_N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}.$$

This implies

$$|S_N| \geq \frac{|z|^{N+1} - 1}{|1 - z|} \to \infty \qquad \text{as} \qquad N \nearrow \infty.$$

Part of the objective of this document is to fully characterize the geometric series considering also the sequence of partial sums when $|z| = 1$.

In response to Daniel's question Katherine Booth constructed a family of series for which the complementary region

$$\Omega_B = \left\{ z \in \mathbb{C}\backslash\overline{D_R(0)} : \liminf_{N\to\infty} \left| \sum_{n=0}^{N} a_n z^n \right| < \infty \right\}$$

is nonempty. The radius of convergence for these examples is $R = 1$ and the set Ω_B consists of precisely one point $x > 1$ on the real axis as indicated in Figure 1. I will now try to explain why this is the case. These series are constructed in more detail in a previous document on *The Domain of Divergence for Complex Power Series,* but to summarize the construction one can take points $a_0, a_1, a_2 \in \mathbb{S}^1$ with

$$a_0 + a_1 x + z_2 x^2 = 0 \tag{2}$$

for some real $x > 1$. Then the series

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_{3k} = a_0$, $a_{3k+1} = a_1$, and $a_{3k+2} = a_2$ for $k = 0, 1, 2, 3, \ldots$ clearly has

$$S_{3k+2}(x) = \sum_{n=0}^{3k+2} a_n x^n = 0.$$

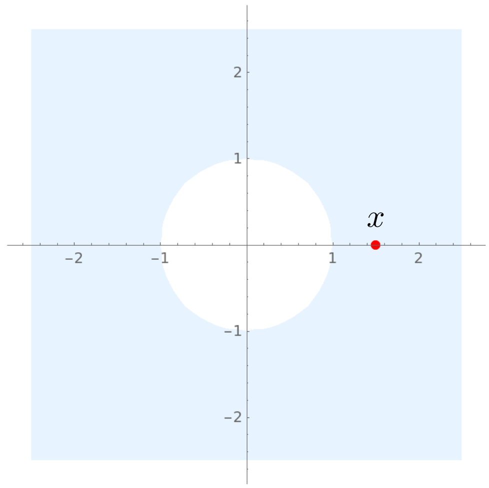


Figure 1: The Savio region (shaded blue outide the unit disk) for a Booth series with a single isolated Booth point $x \in \mathbb{R}$ (indicated in red).

It turns out that given any a_0 with $|a_0| = 1$, the relation (2) determines values of a_1 and a_2 as a function of x for x in a particular real interval

$$\frac{-1+\sqrt{5}}{2} \leq x \leq \frac{1+\sqrt{5}}{2}.$$

Thus one obtains examples with $x \in \Omega_B$ and $\Omega_S \neq \mathbb{C}\backslash\overline{D_R(0)}$. We observe, furthermore that for $|z| > 1$,

$$\sum_{n=0}^{\infty} a_n z^n = (a_0 + a_1 z + a_2 z^2) \sum_{k=0}^{\infty} z^k.$$

The quadratic polynomial

$$p(z) = a_2 z^2 + a_1 z + a_0 = a_2(z - x)\left(z - \frac{a_0}{a_2 x}\right)$$

has roots

$$x \qquad \text{and} \qquad \frac{a_0}{a_2 x}.$$

Notice that the second root lies inside $D_1(0)$ since

$$\left|\frac{a_0}{a_2 x}\right| = \frac{1}{x} < 1.$$

In particular, if $|z| > 1$ but $z \neq x$, then $Z = a_0 + a_1 z + a_2 z^2$ is a fixed nonzero constant, and

$$\sum_{n=0}^{\infty} a_n z^n = (a_0 + a_1 z + a_2 z^2) \sum_{\ell=0}^{\infty} z^{\ell}.$$

shares the divergence behavior of the geometric series. More precisely,

$$S_N = \sum_{n=0}^{N} a_n z^n = \begin{cases} (a_0 + a_1 z + a_2 z^2) \displaystyle\sum_{\ell=0}^{K} z^{\ell}, & N = 3K \\ (a_0 + a_1 z + a_2 z^2) \displaystyle\sum_{\ell=0}^{K} z^{\ell} + a_0 z^{3K+1}, & N = 3K+1 \\ (a_0 + a_1 z + a_2 z^2) \displaystyle\sum_{\ell=0}^{K} z^{\ell} + a_0 z^{3K+1} + a_1 z^{3K+2}, & N = 3K+2. \end{cases}$$

We have, furthermore,

$$\sum_{\ell=0}^{K} z^{\ell} = \frac{1 - z^{K+1}}{1 - z}.$$

Therefore, when $N = 3K$, we have

$$|S_{3K}| = |Z| \left| \frac{1 - z^{K+1}}{1 - z} \right| > |Z| \frac{|z|^{K+1} - 1}{|1 - z|} \to \infty.$$

When $N = 3K + 1$ and for K large

$$\begin{aligned} |S_{3K+1}| &= \left| Z \frac{1 - z^{K+1}}{1 - z} + a_0 z^{3K+1} \right| \\ &\geq |a_0 z^{3K+1}| - |Z| \frac{|1 - z^{K+1}|}{|1 - z|} \\ &> |z|^{3K+1} - 2|Z| \frac{|z|^{K+1}}{|1 - z|} \\ &= |z|^{K+1} \left(|z|^{2k} - \frac{2|Z|}{|1 - z|} \right) \to \infty. \end{aligned}$$

Similarly, when $N = 3K + 2$ and for K large

$$\begin{aligned}
|S_{3K+2}| &= \left| Z\frac{1 - z^{K+1}}{1 - z} + a_0 z^{3K+1} + a_1 z^{3K+2} \right| \\
&\geq |a_2 z^{3K+2}| - |a_0 z^{3K+1}| - |Z|\frac{|1 - z^{K+1}|}{|1 - z|} \\
&> |z|^{3K+2} - |z|^{3K+1} - 2|Z|\frac{|z|^{K+1}}{|1 - z|} \\
&= |z|^{3K+2}\left(1 - \frac{1}{|z|} - \frac{2|Z|}{|1 - z|}\,\frac{1}{|z|^{2K+1}}\right) \to \infty
\end{aligned}$$

since $1 - 1/|z| > 0$ and

$$\frac{2|Z|}{|1 - z|}\,\frac{1}{|z|^{2K+1}} \to 0 \qquad \text{as} \qquad K \nearrow \infty.$$

We have established that for these series $\Omega_B = \{x\}$. We note that Ω_S is open in this case, but we have no general proof that Ω_S is always open and/or nonempty.

Boundary Behavior

Some of the ideas above extend naturally to the boundary of the disk of convergence where $|z| = R$. In this case, there are more possibilities. There can still be convergence of the partial sums to ∞ (Savio points) and divergence with a subsequence of partial sums remaining bounded (Booth points), but there may also be points of convergence and even absolute convergence. Among all these may be further distinctions. First of all, let us call a point with $|z| = R$ for which the series

$$\sum_{n=0}^{\infty} a_n z^n \tag{3}$$

with radius of convergence R converges a **Liebniz** point. The value of the convergent series at a Leibniz point is given by Abel's theorem in terms of the holomorphic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defined by the series on the open disk $D_R(0)$. We have discussed the proof of Abel's limit point theorem elsewhere, but let me state a version suitable for the present context:

Theorem 1 (Abel limit point theorem) *Given a formal series (3) with radius of convergence $R > 0$ and a (Liebniz) point z with $|z| = R$ for which the series converges to a complex number w, we have*

$$w = \lim_{t \nearrow 1} f(t\, z) = \lim_{t \nearrow 1} \sum_{n=0}^{\infty} a_n t^n z^n. \tag{4}$$

Part of the assertion of the theorem, of course, is that the limit appearing in (4) exists. It turns out that this limit

$$\alpha = \lim_{t \nearrow 1} f(t\, z) = \lim_{t \nearrow 1} \sum_{n=0}^{\infty} a_n t^n z^n \tag{5}$$

can exist sometimes **even when the series is not convergent** at $z \in \partial D_R(0)$. In this case, we say the series is **Abel summable** and take the limit $\alpha \in \mathbb{C}$ in (5) as the **Abel sum**. For example, the geometric series defining $g : D_1(0) \to \mathbb{C}$ by

$$g(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \tag{6}$$

evaluated at $z = -1$ becomes

$$1 - 1 + 1 - 1 + \ldots$$

which does not converge. In fact, notice that the partial sums are $1, 0, 1, 0, \ldots$, so this is a Booth (boundary) point for the geometric series. Nevertheless, but for $-1 < x < 1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

so $\alpha = \lim_{x \searrow -1} g(x) = 1/2$, so we say the formal series

$$\sum_{n=0}^{\infty} (-1)^n$$

is Abel summable with Abel sum $\alpha = 1/2$. I think most attention to Abel summability has been given to cases like this where a finite value is assigned to a divergent series. I don't know much about cases where the Abel limit does not exist (as a complex number). Presumably, the values $f(t\, z)$, when the Abel limit does not exist, may either tend to infinity (in the Riemann sphere) or maintain bounded values for t arbitrarily close to 1. Here is a conjecture one might be able to prove:

Conjecture 1 *If z is a Savio (boundary) point for a formal power series, then the Abel limit is also infinity.*

The Geometric Series

For the geometric series, it is evident from (6) that the Abel limit

$$\alpha = \lim_{t \nearrow 1} g(tz)$$

exists for every $z \in \mathbb{S}^1$ except $z = 1$ and takes the value

$$\alpha(z) = \lim_{t \nearrow 1} g(tz) = \frac{1}{1-z}, \qquad z \neq 1. \tag{7}$$

Thus, the series is Abel summable at all points in the circle (except $z = 1$). We already noted that the series does not converge at the antipodal points $z = \pm 1$ with the left point being a Booth point and the right point being a Savio point. Let us see if we can characterize the boundary behavior of the geometric series at the other points with $|z| = 1$.

This is actually pretty easy becuase we still have an explicit formula for the partial sums:

$$S_N = \sum_{n=0}^{N} e^{in\theta} = \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} \tag{8}$$

$\theta \neq 2\pi k$, $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Clearly then

$$|S_N| \leq \frac{2}{1 - e^{i\theta}} < \infty,$$

so none of these points are Savio (boundary) points. On the other hand,

$$\left| e^{i(k+1)\theta} - e^{ik\theta} \right| = \left| 1 - e^{i\theta} \right| > 0$$

when $\theta \neq 2\pi k$, $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. This means the sequence of partial sums cannot be Cauchy. In fact,

$$S_N - S_{N-1} = e^{iN\theta}$$

has modulus 1 (which is no surprise since each term itself is of modulus 1, that is to say, the series obviously fails the basic necessary condition for convergence). The point is that none of these points are Liebniz points either. They are all Booth points.

This situation is illustrated in Figure 2.

The last thing I would like to do is try to see if there is any relation between the Abel limit corresponding to a Booth point $z \in \mathbb{S}^1 \backslash \{1\}$ for the geometric series and

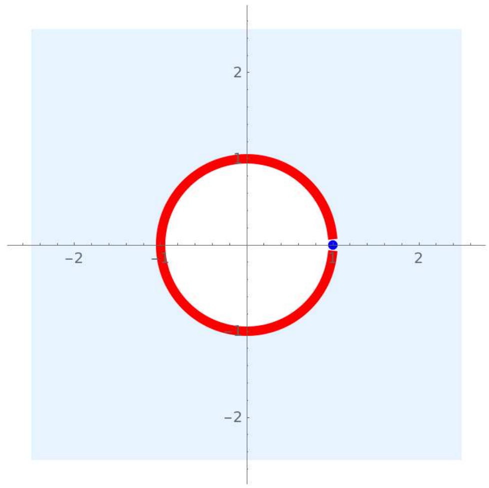


Figure 2: The Savio region (shaded blue outide the unit disk) for the geometric series with a single Savio point $z = 1$ in $\mathbb{S}^1$ (indicated in blue) and all remaining boundary points Booth points (indicated in red).

the sequence of partial sums. Note that for $z = -1$, the Abel limit $\alpha = g(-1) = 1/2$ is the average value of the alternating partial sums $1, 0, 1, 0, \ldots$. Likewise, for the Savio point $z = 1$, one can say the Abel limit is the limit of the average values

$$A_k = \frac{S_0 + S_1 + S_2 + S_3 + \ldots + S_k}{k+1}$$

of the partial sums. I vaguely remember reading about some theorem in either Rudin's *Principles of Mathematical Analysis* or Körner's *Fourier Analysis* relating some kind of averages to something like Abel sums. In any case, let's see what we can say (or see).

First of all for $z = e^{i\theta}$ the Abel sum is

$$\alpha = \frac{1}{1-z} = \frac{1}{1-e^{i\theta}} = \frac{1 - \cos\theta + i\sin\theta}{2(1-\cos\theta)} = \frac{1}{2} + i\,\frac{\sin\theta}{2(1-\cos\theta)}.$$

It is interesting that the real part is always $1/2$ and the imaginary part always takes the sign of $\operatorname{Im} z$. The partial sum formula (8) can also be simplified as

$$S_N = \frac{1 - \cos\theta - \cos[(N+1)\theta] + \cos(N\theta)}{2(1-\cos\theta)} + i\,\frac{\sin\theta - \sin[(N+1)\theta] + \sin(N\theta)}{2(1-\cos\theta)}.$$

Let us consider as a first case $z = i = e^{i\pi/2}$. The partial sums are seen to repeat a cycle of length four in accord with the powers of i:

$$1, 1 + i, i, 0, 1, \ldots$$

These are (consecutively and in counterclockwise order) the points at the corners of the unit square in the first quadrant as indicated in red in Figure 3. The sequence of averages A_k looks like

$$1,\ 1 + \frac{1}{2}i,\ \frac{2}{3}(1+i),\ \frac{1}{2}(1+i),\ \frac{3}{5} + \frac{2}{5}i,\ \frac{2}{3} + \frac{1}{2}i,\ \frac{4}{7} + \frac{4}{7}i,\ \frac{1}{2}(1+i),\ \ldots$$

I did not prove it, but it seems clear that the averages of the partial sums converge

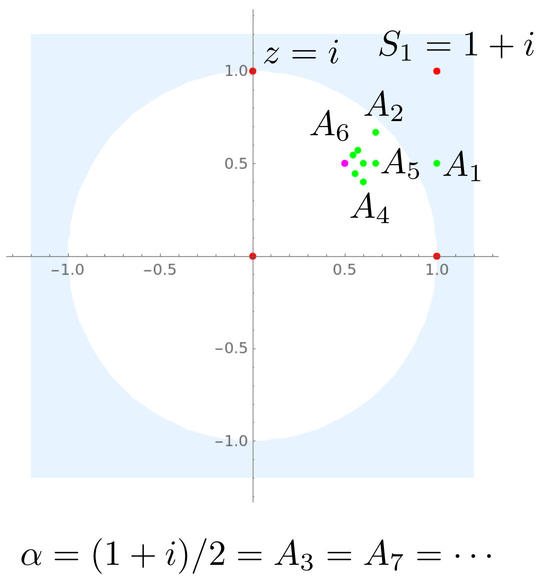
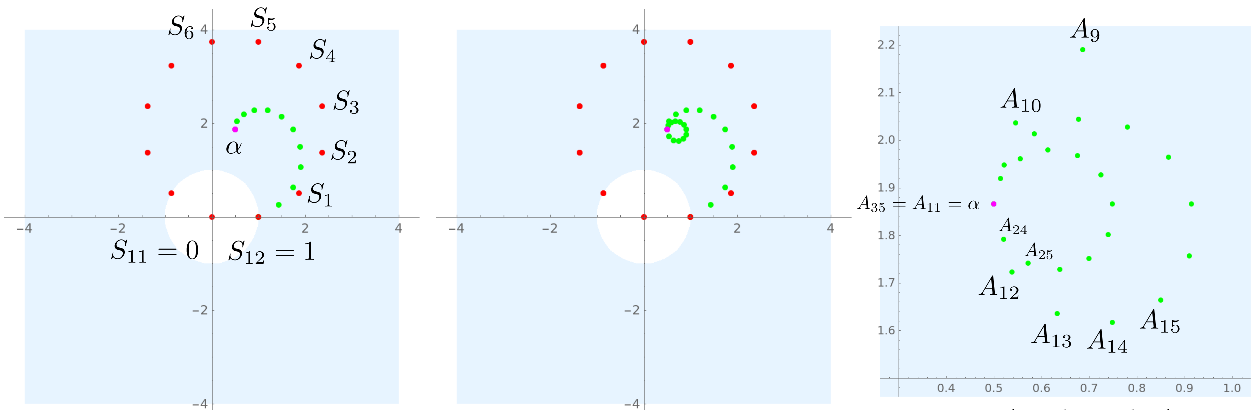


Figure 3: The geometric series $\sum i^n$. Here the sequence of partial sums is indicated in red starting from $S_0 = 1$. The consecutive averages A_k are plotted in green, and the Abel limit α is plotted in magenta.

to the Abel limit of $(1+i)/2$ which is also the center of the square determined by the cycle of the partial sums.

The plot for $z = -i$ looks like the conjugate of the plot in Figure 3, as should be expected with Abel sum $(1-i)/2$. Figure 4 shows the first few terms of the partial



$\theta - \pi/6$, $\alpha = [1 + (2 + \sqrt{3})i]/2$ $\qquad$ A_{12} through A_{23} $\qquad$ A_{24} through A_{35}

Figure 4: The geometric series $\sum z^n$ with $z = e^{i\theta}$, $\theta = \pi/6$. The sequence of partial sums is indicated in red starting from $S_0 = 1$. The consecutive averages A_k are plotted in green, and the Abel limit α is plotted in magenta.

sums and the averages as well as the Abel limit $\alpha = [1 + (2 + \sqrt{3})i]/2$ when $\theta = \pi/6$.

In the examples we have considered with argument a rational multiple of π, the partial sums S_k apparently cycle through the the origin as well as $S_0 = 1$. It appears also that the averages A_k cycle through the Abel sum and converge to it. We shall consider last an example with $z = e^{i\theta}$ where θ is not a rational multiple of π, namely $\theta = 1$. See Figure 5.

Motivated by these examples, the following can presumably be verified.

Conjecture 2 *For $z = e^{i\theta} \in \mathbb{S}^1 \backslash \{1\}$ the sequence of partial sums for the geometric series*

$$\sum_{n=0}^{\infty} z^n$$

lies on a circle of radius

$$\left| \frac{1}{2} + \frac{\sin\theta}{2(1 - \cos\theta)}\ i - 1 \right| = \frac{1}{2|\sin(\theta/2)|}$$

with center the Abel sum

$$\alpha = \frac{1}{2} + \frac{\sin\theta}{2(1 - \cos\theta)}\ i.$$

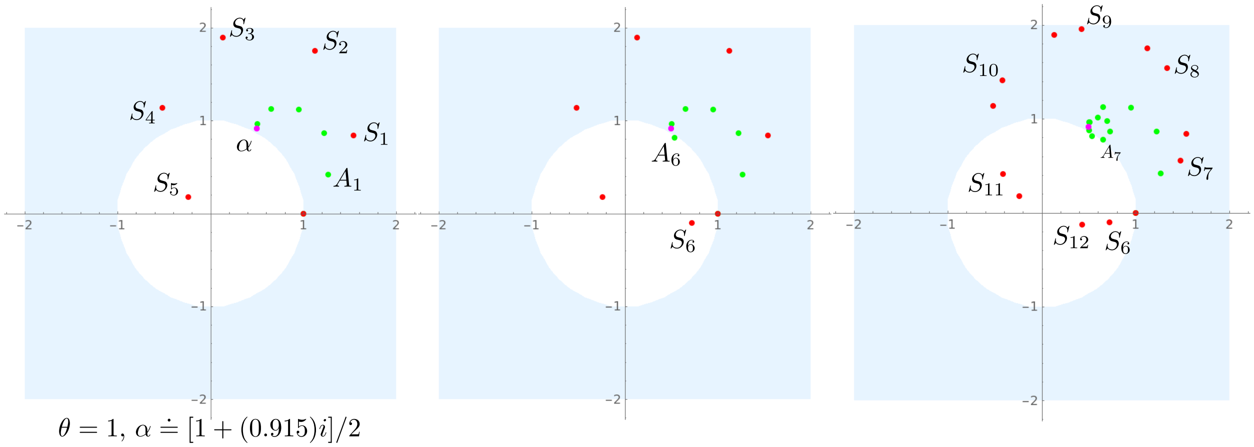


Figure 5: The geometric series $\sum e^{ni} = \sum(\cos n + i \sin n)$ corresponding to argument $\theta = 1$ (radian). The sequence of partial sums is indicated in red starting from $S_0 = 1$. The consecutive averages A_k are plotted in green, and the Abel limit α is plotted in magenta.

and the Abel sum is the limit of the consecutive averages

$$\alpha = \lim_{k\to\infty} A_k, \qquad A_k = \frac{1}{k+1}\sum_{N=0}^{k} S_N.$$

This seems to essentially completely characterize the behavior of the geometric series. (At least I can't think of any other questions to ask about it.)