Roots of a Quadratic Polynomial

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We wish to generalize here the elementary picture indicated in Figure 1. The

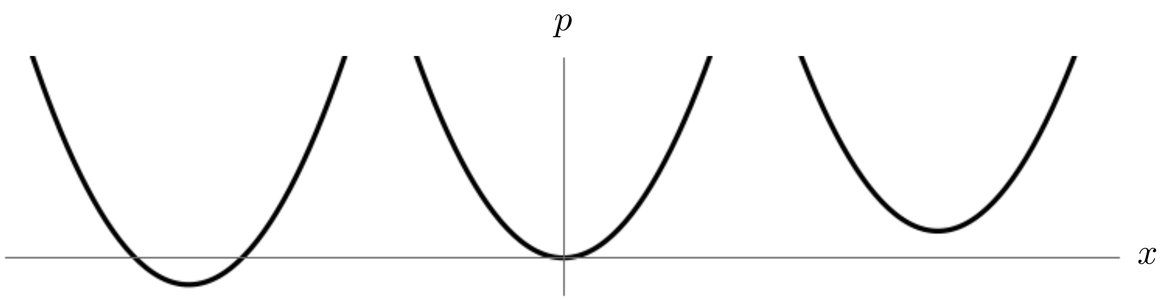

Figure 1: Plots of three quadratic polynomials on the same pair of real axes.

plots show the values of three polynomials $p(x) = a_2x^2 + a_1x + a_0$ and the roots, or solutions of the equation $p(x) = 0$, nicely indicating three distinct cases: (1) two real roots, (2) a single real root of multiplicity two, and (3) no real roots. The illustration is limited in application to polynomials with real coefficients, and it does not illustrate anything about the roots when they are complex valued. It is our objective to illustrate in terms of complex mappings, and specifically a homotopy construction of Dan Romik (employed to give a topological proof of the fundamental theorem of algebra), the geometric location and necessity of roots in the complex plane of a quadratic polynomial $p(z) = a_2z^2 + a_1z + a_0$ with possibly complex coefficients.

The discussion may also be considered as related to Ahlfors' Exercise 1.2.4. In fact, from the point of view of a student of complex analysis, or from the point of view of writing an exposition of complex analysis, the discussion below may be considered as a kind of alternative to Exercise 1.2.4 of Ahlfors. Both are rather involved if all the details are considered. I think the discussion below may be somewhat more instructive overall than careful consideration of the complex square root in the quadratic formula.

Romik's idea is relatively straightforward. The function p represents an entire holomorphic function mapping the complex plane into the complex plane. Apart from the origin $0 \in \mathbb{C}$, the plane can be partitioned by circles

$$\partial D_r(0) = \{z \in \mathbb{C} : |z| < r\}$$

centered at 0 and having radius $r > 0$. Specifically, the punctured plane may be continuously "swept out" by these circles. If the image of each circle $\partial D_r(0)$ under the polynomial p can be determined, plotted, and understood, one should be able to see each root of p as a point z in a specific circle with image $p(z) = 0$. More precisely, we the mapping $h(t, r) = re^{it}$ may be considered as a smooth deformation, or homotopy, of any one of these circles to any other giving rise to a homotopy of the images $H(t, r) = p(re^{it})$. It is our basic objective to understand these image curves

$$\{p(re^{it}) : 0 \le t \le 2\pi\}$$

in the complex plane and determine which among them pass through the origin, or at least that one of them must. There is, of course, the exceptional case $p(z) = a_2 z^2$ in which there is a single root of multiplicity two at $z = 0$.

Let us say for a moment $p(z) = z^2$ corresponding to the polynomial with real coefficients plotted in the middle in Figure 1. Circles

$$\gamma(t) = re^{it}$$

with radii $j/16$, $j = 1, 2, 3, \ldots, 28$ are plotted on the left in Figure 2 along with the image circles

$$p \circ \gamma(t) = r^2 e^{2it}$$

under the polynomial p.

Each image circle is covered twice. This can be more clearly seen if we plot the image in the Riemann surface $\mathcal{R}$ for z^2. We have done this is Figure 3

where $\mathcal{R}$ is represented by two planes each of which has a branch cut along the negative real axis.

1 Preliminaries

The assumption that the polynomial p is of degree 2 or quadratic means the coefficient a_2 is nonzero. Thus, we can write

$$p(z) = a_2[z^2 + (a_1/a_2)z + (a_0/a_2)]$$

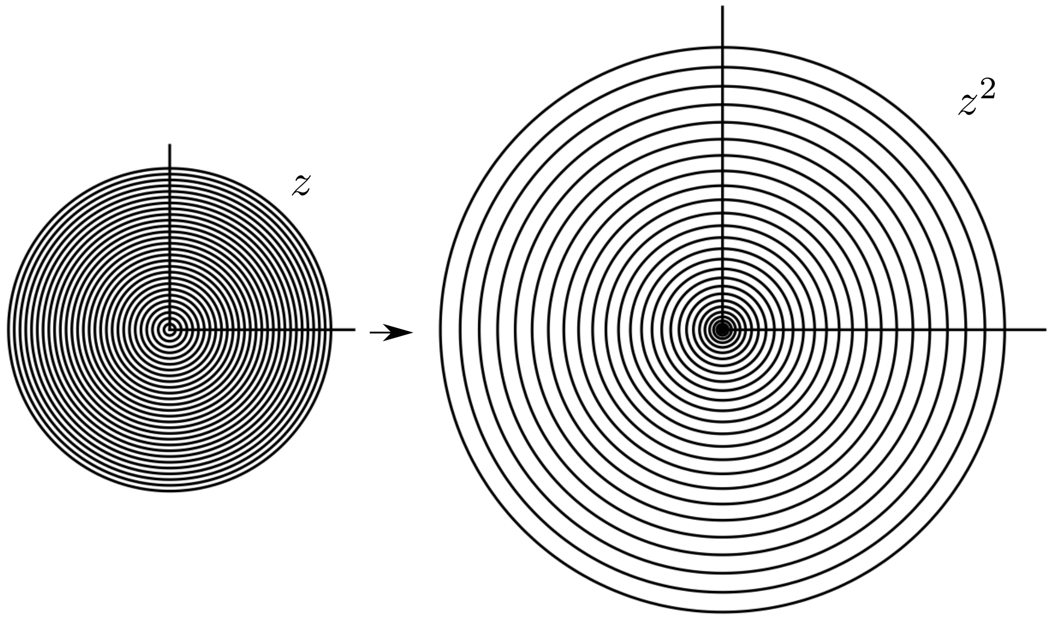

Figure 2: The mapping $p(z) = z^2$. The root is at $z = 0$, so none of the image circles passes through $z^2 = 0$ in the image. In cases where there is a nonzero root, that root should lie on a circle with center at $z = 0$ and the image curve should pass through $p(z) = 0$ in the codomain.

where $\tilde{a}_1 = a_1/a_2$ and $\tilde{a}_0 = a_0/a_2$ are well-defined complex numbers. Recall that multiplication by a nonzero complex number a_2 corresponds to a rotation of $\mathbb{C}$ by an angle $\mathrm{Arg}(a_2)$ and a dilation of $\mathbb{C}$ by $|a_2|$ (in either order). Thus, the image of each circle $\partial D_r(0)$ under $p(z) = a_2 z^2 + a_1 z + a_0$ is a fixed rotation/dilation of the image of that circle under

$$\tilde{p} = z^2 + \tilde{a}_1 z + \tilde{a}_0.$$

The same applies of course to the image of $0 \in \mathbb{C}$. In short, we may henceforth consider the polynomial $p(z) = z^2 + a_1 z + a_0$ to be **monic** having leading coefficient $a_2 = 1$. We need only keep in mind that a final rotation may be necessary in order to accurately represent the non-monic case. The rotation just discussed is illustrated for a polynomial $p(z) = a_2 z^2$ in Figure 4.

To complete our preliminary considerations, we complete the square in the monic

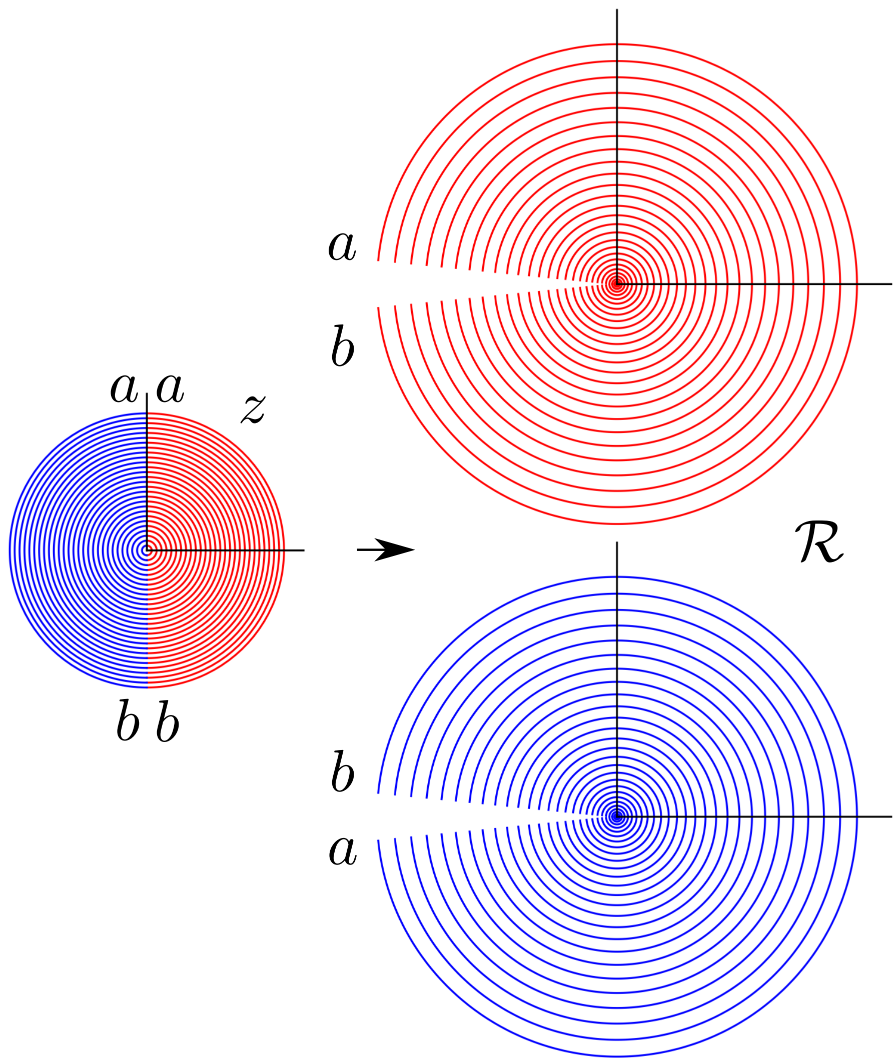

Figure 3: The mapping $p : \mathbb{C} \to \mathcal{R}$ by $p(z) = z^2$. Here we have taken as codomain the Riemann surface $\mathcal{R}$ with two sheets and a single branch point at $z^2 = 0$.

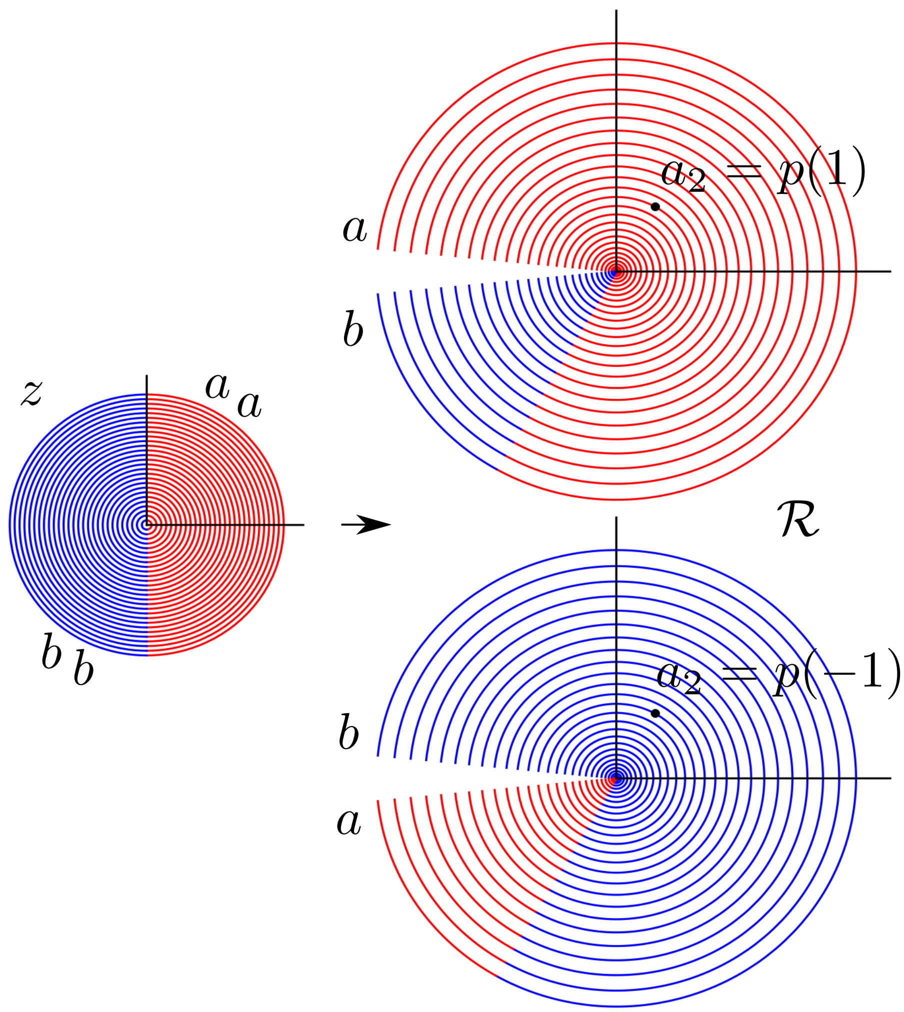

Figure 4: The mapping $p : \mathbb{C} \to \mathcal{R}$ by $p(z) = a_2 z^2$. Here we have taken $a_2 = e^{i\pi/3}$. There is no dilation, but only a rotation in the Riemann surface.

polynomial:

$$p(z) = \left(z - \frac{a_1}{2}\right)^2 + a_0 - \frac{a_1^2}{4}. \tag{1}$$

The quantity

$$b_0 = a_0 - \frac{a_1^2}{4} = \frac{4a_0 - a_1^2}{4}$$

contains the familiar descriminant $a_1^2 - 4a_0$. When a_0 and a_0 are real, the sign of this quantity determines which of the cases in Figure 1 occurs. Naturally, when $a_1^2 - 4a_0$ is complex there can be no distinguishing inequality determining the sign of the quantity. Nevertheless, the condition $b_0 = 0$ plays a natural role in distinguishing cases. In particular, we note that when $b_0 = 0$ there is a single (complex) root

$$z = \frac{a_1}{2}$$

of multiplicity two, and this root will be real precisely when $a_1 \in \mathbb{R}$ which, by the vanishing of b_0, implies $a_0 = 0$ as well. In view of this comment we should expect something interesting to happen (more interesting than the cases considered in Figures 2-4) when there is a nonzero root of multiplicity two. Note that this feature is absent from the illustration of Figure 1: The geometry of the illustration when $p(x) = x^2$ and when $p(x) = (x - a_1/2)^2$ is essentially the same.

In view of the expression (1) we can also characterize the case of Figure 1 in which there are two distinct real roots as corresponding to the case in which b_0 is real and nonzero and a_1 is real. Thus, we will wish to pay specific attention to this case with respect to Romik's construction/illustration.

Naturally, we will also seek to distinguish (or see) the case when the roots are distinct but complex conjugates. Here is an overall outline of the cases I plan to consider.

Case 0 $a_0 = 0$

- **Case 0.1** $a_0 = 0 = b_0$ (In this case $a_1 = 0$, and I've already considered it above.)
- **Case 0.2** $a_0 = 0$, $b_0 \neq 0$ (In this case $a_1 \neq 0$.)

Case 1 $b_0 = 0$

Case 2 $b_0 \neq 0$.

Notice that **Case 0.1** may be considered as a special case of, and included in, **Case 1**. Similarly, **Case 0.2** may be considered as a special case of, and included in, **Case 2**.

2 A Real Root at $z = 0$

In **Case 0.2** we may write

$$p(z) = \frac{a_1^2}{4}\left[\left(\frac{2}{a_1}z + 1\right)^2 - 1\right] \tag{2}$$

and isolate the interesting mapping

$$q(z) = \left(\frac{2}{a_1}z + 1\right)^2 = (c_1 z + 1)^2 \tag{3}$$

with $c_1 = 2/a_1 \neq 0$. The mapping of circles under q turns out to be fundamental to our objective in all cases, so we consider it in some detail here.

Noting that the rotation/dilation due to the factor $a_1^2/4$ in (2) does not contribute materially to the geometry, we can consider directly the image of Romik's domain circles under the mapping

$$\phi(z) = (c_1 z + 1)^2 - 1 = q(z) - 1.$$

Obviously, ϕ is simply a shift left by 1 of the image

$$q(\partial D_r(0)) = \{q(re^{it}) : 0 \leq t \leq 2\pi\}$$

determined by q, so we consider that image. Finally, the rotation/dilation associated with multiplication by c_1 in (3) maps a circle $\partial D_r(0)$ to the circle $\partial D_{|c_1|r}(0)$ in the same family with the ordering of the radii preserved. As a consequence, it is natural to consider, at least for the purposes of initial computation and illustration, the special case $c_1 = |c_1| \in \mathbb{R}$ and $q(z) = (|c_1|z + 1)^2$.

Let us begin with a computation (assuming $c_1 = |c_1| > 0$):

$$\begin{aligned} q(re^{it}) &= (1 + |c_1|re^{it})^2 \\ &= 1 + 2|c_1|re^{it} + |c_1|^2 r^2 e^{2it} \\ &= 1 + 2|c_1|r\cos t + |c_1|^2 r^2 \cos(2t) + i[2|c_1|r\sin t + |c_1|^2 r^2 \sin(2t)]. \end{aligned}$$

At least three cases are worth distinguishing:

(i) $r < 1/|c_1|$. In this case, the circle

$$C_1(r) = \{c_1 z + 1 : z \in \partial D_r(0)\} \tag{4}$$

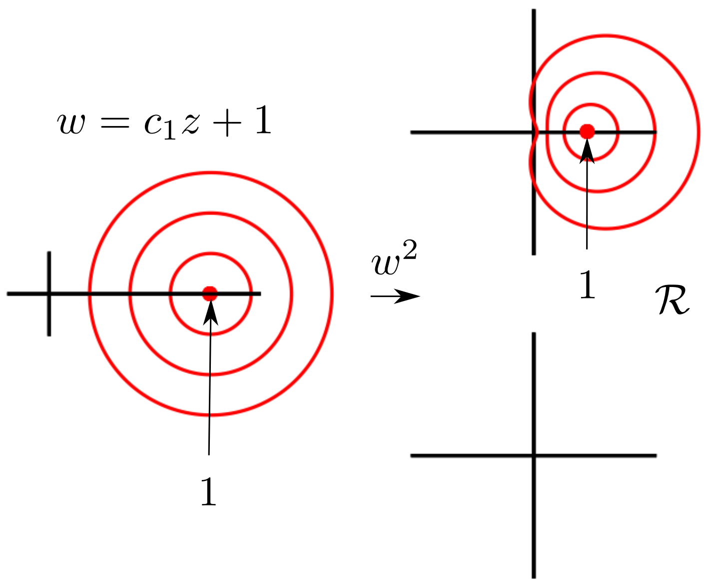

Figure 5: The image of a circle under $q(z) = (c_1 z + 1)^2$. The domain on the left is not the domain of the polynomial $p = p(z)$, but rather the image of that domain under the affine transformation $w = c_1 z + 1$ where we have assumed $c_1 = |c_1| > 0$ and $r < 1/|c_1|$. It may be noted also that different scales are used in the w (complex) plane and the Riemann surface $\mathcal{R}$ for w^2 as can be seen from the relative size on $w = 1$ (on the left) compared to $w^2 = 1$ (on the right). Finally, it should be noted that the image $q(\partial D_r(0))$ shown here is not the same as the image $p(\partial D_r(0))$, but the image $p(\partial D_r(0))$ is obtained by a left shift by $1 \in \mathbb{C}$. Thus, the image $p(\partial D_r(0))$ with $r < 1/|c_1|$ in **Case 0.2** can never pass through $p(z) = 0$; no circle $\partial D_r(0)$ with $r < 1/|c_1|$ can contain a root of p.

is a circle of radius $|c_1|r$ with center $w = 1$ entirely contained in the open right half plane as indicated on the left in Figure 5. Consequently, the entire image $q(\partial D_r(0))$ lies in the first sheet of the Riemann surface $\mathcal{R}$. Various properties may be verified about the important image $q(\partial D_r(0))$, and we will verify several of them. It is easy to see immediately that in this particular case $q(\partial D_r(0))$ is a simple closed curve intersecting the real axis exactly twice in points $x_1(r)$ and $x_2(r)$ satisfying

$$0 < x_1(r) < 1 < x_2(r).$$

The intersection of $q(\partial D_r(0))$ with the real axis is, moreover, transverse and in fact, the image curve intersects the real axis at right angles. It is also immediate that $x_1(r)$ is a decreasing function of r for $0 < r < 1/|c_1|$ with

$$\lim_{r \nearrow 1/|c_1|} x_1(r) = 0,$$

and $x_2(r)$ is an increasing function of r on the same interval with

$$\lim_{r \nearrow 1/|c_1|} x_2(r) = 4.$$

Naturally,

$$\lim_{r \searrow 0} x_j(r) = 1 \qquad \text{for } j = 1, 2.$$

In particular, the simple closed curve $q(\partial D_r(0))$ with $r < 1/|c_1|$ does not encircle $0 \in \mathbb{C}$. The simple closed curve $p(\partial D_r(0))$ on the other hand **always** goes around $0 \in \mathbb{C}$ in **Case 0.2(i)** when $r < 1/|c_1|$.

(ii) $r = 1/|c_1|$. As indicated in Figure 6, the images $q(\partial D_r(0))$ and $p(\partial D_r(0))$ cease to be smooth curves when $r = 1/|c_1|$ and the circle C_1 defined in (4) passes through the pre-image of the branch point in $\mathcal{R}$. In this case, the non-conformality of the square introduces a cusp into the image $q(\partial D_r(0))$ at $0 \in \mathcal{R}$. Aside from the appearance of this singularity, the image still retains many of the characteristics described in the previous **Case 0.2(i)**. In particular, $q(\partial D_r(0))$ is a simple closed curve which contains no points on the negative real axis: There is a cusp point at $x_1 = 0$ and a point $x_2 = 4$ at which the curve crosses the positive real axis at a right angle. The left translation $p(\partial D_r(0))$ in the case $r = 1/|c_1|$ is a simple closed curve around the origin, and there is no zero in $\partial D_r(0)$ when $r = 1/|c_1|$. Also, the images $q(\partial D_r(0))$ and $p(\partial D_r(0))$ still lie in the first sheet of $\mathcal{R}$.

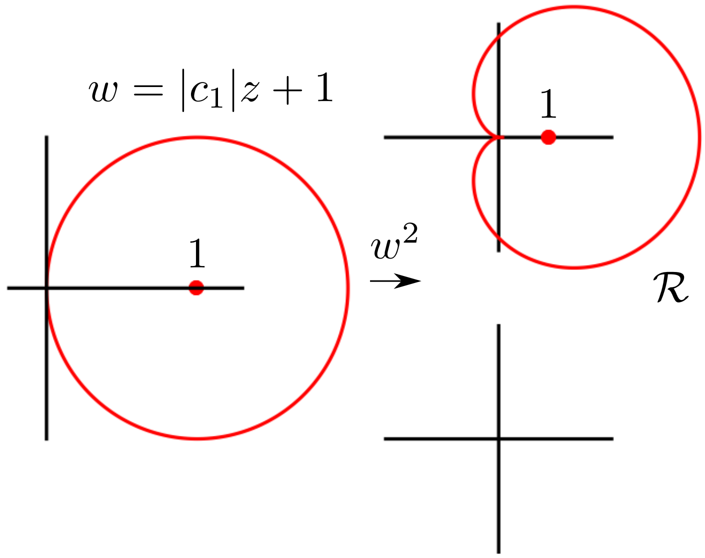

Figure 6: The image of a circle of radius $r = 1/|c_1|$ under $q(z) = (c_1 z + 1)^2$. Still the image $p(\partial D_r(0))$ with $r = 1/|c_1|$ in **Case 0.2(ii)** does not pass through $p(z) = 0$; no circle $\partial D_r(0)$ with $r = 1/|c_1|$ can contain a root of p.

(iii) $r > 1/|c_1|$. In this case, the circle C_1 defined in (4) passes through the left half plane, and the image $q(\partial D_r(0))$ in $\mathcal{R}$ enters the second sheet of the Riemann surface. The image $q(\partial D_r(0))$ furthermore crosses the real axis **four times** at three distinct points. There are two perpendicular crossings at

$$x_1(r) = q \circ \gamma(0) = q(r) = 1 + 2|c_1|r + |c_1|^2 r^2 = (1 + |c_1|r)^2 > 1, \text{ and}$$

$$x_2(r) = q \circ \gamma(\pi) = q(-r) = 1 - 2|c_1|r + |c_1|r^2 = (1 - |c_1|r)^2 > 0.$$

We also have the inequality

$$x_2(r) < x_1(r),$$

and we note that $1 - |c_1|r < 0$. The situation is illustrated in Figure 7. There are also two crossings at the same point on the negative real axis corresponding to the two points where C_1 crosses the imaginary axis.

Since $r > 1/|c_1|$, it is easy to see $x_2(r)$ is increasing and satisfies

$$\lim_{r \nearrow \infty} x_2(r) = +\infty.$$

Consequently, there is exactly one circle $\partial D_r(0)$ with a point mapping under q to $w^2 = 1$ and hence containing the other root $z = -a_1$ of the equation $p(z) = z(z + a_1) = 0$. It is easy enough to find the radius $r > 1/|c_1|$ and the point $w \in C_1$ corresponding to the root. We need in fact,

$$1 - |c_1|r = -1 \qquad \text{or} \qquad r = \frac{2}{|c_1|} = |a_1|.$$

For the actual root in $\partial D_r(0)$ we need

$$c_1 z + 1 = \frac{2}{a_1} z + 1 = -1 \qquad \text{or} \qquad z = -a_1$$

as expected.

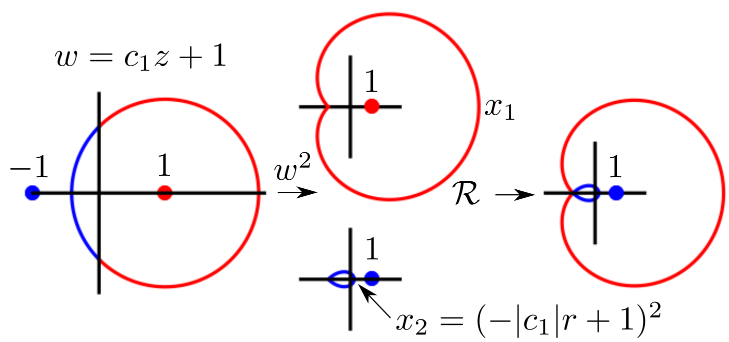

Figure 7: The image of a circle of radius $r > 1/|c_1|$ under $q(z) = (c_1 z + 1)^2$. The image $q(\partial D_r(0))$ in **Case 0.2(iii)** may be viewed as consisting of two distinct loops in $\mathcal{R}$ both of which enclose the branch point at $0 \in \mathbb{C}$; the larger loop (red) encircles also $w^2 = 1$, but the smaller loop may or may not encircle $w^2 = 1$. Precisely one circle $\partial D_r(0)$ with $r > 1/|c_1|$ contains the other root of p. The figure on the right gives the projection of $q(\partial D_r(0))$ in $\mathcal{R}$ into $\mathbb{C}$.

This is all straightforward, but the important and interesting aspect is seen in Figures 5-7. The polynomial q maps small circles ($0 < r \leq 1/|c_1| = |a_1|/2$) to simple closed curves, and the translation to the left by 1 giving p according to (2) produces curves encircling but not passing through the origin. For $r > 1/|c_1| = |a_1|/2$, the quadratic polynomial q maps circles to curves consisting of two loops both of which grow. The inner loop does not initially (for r close to $1/|c_1| = |a_1|/2$) encircle the origin, but the inner loops grow and eventually pass through the origin for $r = |a_1|$ and then enclose the origin for $r > 2/|c_1| = |a_1|$.

A Visualization Problem

As we will see this deformation of a curve in the Riemann surface $\mathcal{R}$ starting from a small simple closed curve near a point and developing a second loop (across the branch point) which eventually crosses and encloses the origin may be viewed as the essential behavior of a complex quadratic mapping. The appearance of the singularity leading to the second loop strongly suggests a desingularization involving a space curve with a vertical point with respect to projection; see Figure 8. It is also easy to produce a

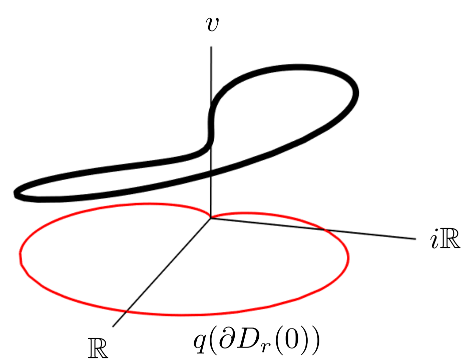

Figure 8: The image of a circle of radius $r = 1/|c_1|$ under $q(z) = (c_1 z+1)^2$. Here $q(\partial D_r(0))$ in **Case 0.2(ii)** is a singular curve with a singularity at the branch point $0 \in \mathcal{R}$ (first sheet). Depicted also is a smooth regular curve in a three dimensional space determined by a "visualization direction" v. The projection of the regular curve is the image $q(\partial D_r(0))$ with a cusp.

family of regular curves in space in which the vertical point "twists" to form a loop projecting to the inner loop of the images $q(\partial D_r(0))$ for $r > 1/|c_1|$.

There are various visualizations of the Riemann surface $\mathcal{R}$ as a singular surface in $\mathbb{R}^3$, and it is usual for these visualizations that the sheets of the surface project orthogonally onto $\mathbb{C}$. It is not usual that a non-singular regular curve with a vertical point like that depicted in Figure 8 can lie within the surface representing $\mathcal{R}$. For one thing, such surfaces usually have a conical character locally at the branch point. One can imagine and alternative visualization surface with a "spread out" branch point so that the surface is helicoidal along a "branch axis." Such a surface can admit a curve with a vertical point at the branch, but it is not (at all) clear that such a surface can admit a curve that "twists" as required to desingularize the homotopy in question. Furthermore, the models of the Riemann surface $\mathcal{R}$ in $\mathbb{R}^3$ all have either self-intersections or identified cuts.

What might be really nice is to give a surface in **four** dimensionas (or possibly more) in which the two sheets of the Riemann surface $\mathcal{R}$ do not self-intersect and come together naturally at the branch point $0 \in \mathbb{C}$. This is definitely possible. One would like to arrange such a singular surface, furthermore, so that it admits the homotopy determined by q on the Romik circles in such a way that each image $q(\partial D_r(0))$ is a regular curve projecting, say with respect to some particular visualization direction $v \in \mathbb{R}^4$ into a (complex) plane giving the image curves we have plotted in Figures 5-7. I do not know if this is possible.

3 Repeated Roots

Here we consider **Case 1** when

$$b_0 = a_0 - \frac{a_1^2}{4} = 0$$

and there is a single root of multiplicity two given by $z = a_1/2 \neq 0$. The equation is trivial to solve, as may even be asserted concerning quadratic equations in all cases, but of course our objective is to see something new and fundamental in the illustration resulting from the Romik construction. We may assume we are in the complement of **Case 0.1**, so that $a_0 \in \mathbb{C}\backslash\{0\}$ and consequently $a_1 \in \mathbb{C}\backslash\{0\}$ as well. We write then

$$p(z) = \left(z + \frac{a_1}{2}\right)^2 = \frac{a_1^2}{4}\left(\frac{2}{a_1}\,z + 1\right)^2 = \frac{a_1^2}{4}\,q(z).$$

Consequently, we see that up to a rotation, the image of p is given by the image of q, which we have just analyzed in the previous section. We are interested then in the condition

$$0 \in q(\partial D_r(0)),$$

and we know this happens exactly once in the case $r = 1/|c_1| = |a_1|/2$ corresponding to the single root $z = a_1/2$.

4 Distinct Roots

Here we consider **Case 2** in which

$$b_0 = a_0 - \frac{a_1}{4} \in \mathbb{C}\backslash\{0\}.$$

We may assume also that we are in the complement of **Case 0.2** discussed above so that $a_0 \in \mathbb{C}\backslash\{0\}$, but it may be the case that $a_1 = 0$. In fact, we consider subcases:

Case 2a $b_0 \neq 0$, but $a_1 = 0$.

In this case, we may dispense with b_0 and write simply

$$p(z) = z^2 + a_0$$

where $a_0 = b_0 \neq 0$. The image $p(\partial D_r(0))$ is always a (double covered) circle of radius r^2 and center a_0. Let us say a_0 is in the first quadrant in $\mathcal{R}$ as indicated in Figure 9. Then $-a_0$ is in the third quadrant, and the principal square root of $-a_0$ is in the fourth quadrant and is a root. The other root is $-\sqrt{-a_0}$.

Case 2b $b_0 \neq 0$ and $a_1 \neq 0$.

Here we write

$$p(z) = \frac{a_1^2}{4}\left[\left(\frac{2}{a_1}\,z + 1\right)^2 + \frac{4b_0}{a_1^2}\right]$$

Ignoring the preliminary dilation/rotation due to the factor $c_1 = 2/a_1$, we understand the mapping $q(z) = (c_1 z + 1)^2$ in terms of its image in the Riemann surface $\mathcal{R}$: For small r the image $q(\partial D_r(0))$ is a (small) simple closed curve encircling $q(0) = 1$. As r increases a cusp develops in the image curve at $w^2 = 0$ corresponding to $r = 1/|c_1|$. For $r > 1/|c_1|$ a singular loop in the first sheet of $\mathcal{R}$ grows and is joined by a second inner loop in the second sheet of $\mathcal{R}$ which also grows. We need now to examine more closely the nature of this growth.

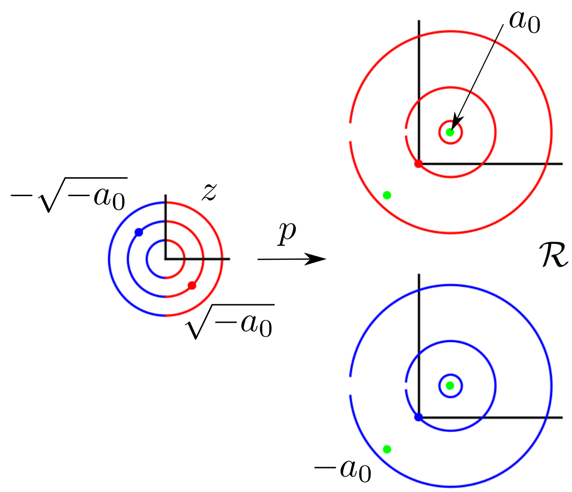

Figure 9: **Case 2a** Mapping of the polynomial $p(z) = z^2 - a_0$. In this particular illustration we have take $a_0 = 1 + i$ and the scales in the domain and codomain match.

Note that the entire image is subjected here to a nontrivial translation by the quantity $4b_0/a_1^2 \in \mathbb{C}\backslash\{0\}$. The point corresponding to the point $q(0) = 1$ which the initial loops for $r < 1/|c_1|$ encircle is

$$\frac{4b_0}{a_1^2} + 1 = \frac{4a_0}{a_1^2} \in \mathbb{C}\backslash\{0\}.$$

Recall that in the complement of **Case 0.2** we have $a_0 \neq 0$.

We consider therefore the homotopy

$$\alpha(t) = \alpha(t;r) = q(re^{it}) + \frac{4b_0}{a_1^2} = (1 + |c_1|re^{it})^2 - 1 + \frac{4a_0}{a_1^2}. \tag{5}$$

Expanding Radial Loops

In view of the expression (5) we see that our (geometric) understanding of quadratic equations with complex coefficients in terms of complex mappings will be complete if we understand the homotopy

$$h(t,r) = \left(|c_1|re^{it} + 1\right)^2 - 1 \tag{6}$$

first considered in Section 2 above. This will take a substantial investment, but hopefully the resulting "big picture" concerning the roots of quadratic equations will be worth it.

Some terminology will be useful. We wish to generalize in various ways the example of an expanding circle $H_0 : [0, 2\pi] \times [0, \infty) \to \mathbb{C}$ given by

$$H_0(\theta, r) = P + re^{i\theta}.$$

We considered various expanding circles with center $P = 0$ in the discussion of $p(z) = z^2$ in the introduction, and we considered an example with center $P = a_0 \neq 0$ in the discussion of **Case 2a** above for $p(z)z^2 + a_0$.

It will be convenient to consider 2π periodic functions $f : \mathbb{R} \to \mathbb{C}$ on a representative interval I of length 2π. As a convention, when we write $f : I \to \mathbb{C}$ and say f is periodic with period 2π, we have in mind the 2π periodic extension of f to the domain $\mathbb{R}$. In such a case, we can freely change the representative interval I and consider, for example, $f : [0, 2\pi] \to \mathbb{C}$ or $f : [-\pi, \pi] \to \mathbb{C}$. More generally, we will consider homotopies similar to H_0 above that are 2π periodic in one variable and use the same convention.

Let us define an **expanding radial loop** with **expansion center** $P \in \mathbb{C}$ to be a continuous homotopy $H : [0, 2\pi] \times [0, \infty) \to \mathbb{C}$ of the form

$$H(\theta, s) = P + R(\theta, s)e^{i\theta}$$

with the following properties:

1. $R : [0, 2\pi] \times [0, \infty) \to [0, \infty)$ is continuous.

2. R is 2π periodic in θ.

3. $R(\theta, 0) = 0$ and R is (strictly) increasing in s.

For $s > 0$ fixed

$$\Gamma_s = \{H(\theta, s) : 0 \leq \theta \leq 2\pi\} \tag{7}$$

is a simple closed curve. Our designation as these curves as **radial loops** has the same meaning as "star shaped" in other contexts. Each image curve Γ_s for $s > 0$ is the boundary of a star shaped domain

$$\Omega_s = \{z \in \mathbb{C} : |z - P| < R(\mathrm{Arg}(z - P), s)\} \tag{8}$$

An expanding radial loop H is said to have an **eventual center** at $Q \in \mathbb{C}$ if for some $T_0 > 0$ there is a continuous homotopy $G : [0, 2\pi] \times [T_0, \infty) \to \mathbb{C}$ of the form

$$G(\theta, t) = Q + M(\theta, t)e^{i\theta}$$

with the following properties:

1. $M : [0, 2\pi] \times [T_0, \infty) \to [0, \infty)$ is continuous.
2. M is 2π periodic in θ.
3. $M(\theta, T_0) > 0$ and M is (strictly) increasing in t,

and for some $S_0 > 0$ the following hold:

(i) There is a continuous one-to-one (strictly) increasing function $\tau : [S_0, \infty) \to [T_0, \infty)$,

(ii) There is a continuous function $\psi : [0, 2\pi] \times [S_0, \infty) \to \mathbb{R}$ such that for fixed $s \geq S_0$

- **(a)** $\psi = \psi(\theta, s)$ is (strictly) increasing and periodic of period 2π in θ and
- **(b)** $\{\psi(\theta, s) : 0 \leq \theta \leq 2\pi\}$ is an interval of length 2π.

(iii) For $s \geq S_0$ we have $H(\theta, s) = G(\psi(\theta, s), \tau(s))$ so that in particular

$$\Gamma_s = \{H(\theta, s) : 0 \leq \theta \leq 2\pi\} = \{G(\theta, \tau(s)) : 0 \leq \theta \leq 2\pi\}. \tag{9}$$

We wish to apply this terminology and these definitions to the homotopy given in (6). It will also be convenient sometimes to write $\rho = |c_1|r > 0$ so that

$$h(t) = h(t, r) = (\rho e^{it} + 1)^2 - 1.$$

Lemma 1 *For $0 < r < 1/|c_1|$, i.e., $\rho < 1$*

$$\{(\rho e^{it} + 1)^2 - 1 : -\pi \leq t \leq \pi\}$$

is a smooth simple closed curve, and there is a function $H : [-\pi, \pi] \times [0, 1) \to \mathbb{C}$ of the form

$$H(\theta, \rho) = R(\theta, \rho)e^{i\theta}$$

such that for $0 \leq \rho \leq 1$

$$\{(\rho e^{it} + 1)^2 - 1 : -\pi \leq t \leq \pi\} = \{R(\theta, \rho)e^{i\theta} : -\pi \leq \theta \leq \pi\}.$$

More precisely, there are continuous functions $R : [-\pi, \pi] \times [0, 1] \to [0, \infty)$ and $\psi : [-\pi, \pi] \times (0, 1] \to \mathbb{R}$ with

(i) *$R = R(\theta, \rho)$ is even in θ (for fixed ρ), strictly increasing in ρ (for fixed θ) and $R(\theta, 0) \equiv 0$. In fact, $R = R(\theta, \rho)$ is smooth for $0 < \rho < 1$ and*

$$\frac{\partial}{\partial R}(\theta, \rho) > 0.$$

(ii) *$\psi = \psi(\theta, \rho)$ is (strictly) increasing, odd, and 2π periodic in θ for $\rho > 0$ fixed with $\psi(0, \rho) = 0$ and $\psi(\pm\pi, \rho) = \pm\pi$.*

(iii) *For $0 \le \rho \le 1$ we have*

$$h(\theta, \rho) = H(\psi(\theta, \rho), \rho),$$

that is

$$\left(\rho e^{it} + 1\right)^2 - 1 = R(\psi(t), \rho) e^{i\psi(t)}. \tag{10}$$

We wish to extend the homotopy H in the lemma above to a full expanding radial loop. We also wish to obtain some more detailed information about the geometry of the loops themselves.

Lemma 2 *Let H be the homotopy defined in Lemma 1. For $0 < \rho \le 1/2$,*

$$\Gamma_\rho = \{H(\theta, \rho) : -\pi \le \theta \le \pi\}$$

is a smooth convex curve with

$$\frac{\partial}{\partial \theta} R(\theta, \rho) < 0 \qquad \text{for} \qquad 0 < \theta < \pi. \tag{11}$$

For $1/2 < \rho < 1$ the inequality (11) still holds and

$$\Gamma_\rho = \{H(\theta, \rho) : -\pi \le \theta \le \pi\}$$

is a smooth curve which is not convex. In this case $\operatorname{Im} H(\theta, \rho)$ satisfies

(i) $\operatorname{Im} H(0, \rho) = \operatorname{Im} H(\pm\pi, \rho) = 0$ *and*

(ii) $\operatorname{Im} H(\theta, \rho)$ *increases for $0 < \theta < \theta_{\max}$ to a positive maximum $\operatorname{Im} H(\theta_{\max}, \rho)$ for some $\theta_{\max} \in (0, \pi)$ and then decreases for $\theta_{\max} < \theta < \pi$.*

The behavior just described for $\operatorname{Im} H(\theta, \rho)$ holds also for $0 < \rho \le 1/2$. For $1/2 < \rho < 1$, however, the real part $\operatorname{Re} H(\theta, \rho)$ satisfies the following:

(i) $\operatorname{Re} H(0, \rho) = \rho(\rho + 2)$, $\operatorname{Re} H(\pm\pi, \rho) = \rho(\rho - 2) < 0$ *and*

(ii) $\operatorname{Re} H(\theta,\rho)$ *decreases for* $0<\theta<\theta_1$ *to a negative minimum* $\operatorname{Re} H(\theta_1,\rho)<\rho(\rho-2)$ *for some* $\theta_1 \in (\theta_{\max},\pi)$ *and then increases for* $\theta_1<\theta<\pi$.

Lemma 3 *Let* H *be the homotopy defined in Lemma 1. For* $\rho=1$,

$$\Gamma_1 = \{H(\theta,1) : -\pi \le \theta \le \pi\}$$

is a singular nonconvex curve with a cusp at $\theta=\pm\pi$. *The curve is smooth elsewhere, and it is still true that*

$$\frac{\partial}{\partial\theta} R(\theta,1) < 0 \qquad \text{for} \qquad 0<\theta<\pi.$$

$\operatorname{Im} H(\theta,1)$ *satisfies*

(i) $\operatorname{Im} H(0,1) = \operatorname{Im} H(\pm\pi,1) = 0$ *and*

(ii) $\operatorname{Im} H(\theta,1)$ *increases for* $0<\theta<\theta_{\max}$ *to a positive maximum* $\operatorname{Im} H(\theta_{\max},1)$ *for some* $\theta_{\max}\in(0,\pi)$ *and then decreases for* $\theta_{\max}<\theta<\pi$.

$\operatorname{Re} H(\theta,\rho)$ *satisfies*

(i) $\operatorname{Re} H(0,1)=3$, $\operatorname{Re} H(\pm\pi,1)=-1$ *and*

(ii) $\operatorname{Re} H(\theta,1)$ *decreases for* $0<\theta<\theta_1$ *to a negative minimum* $\operatorname{Re} H(\theta_1,1)<-1$ *for some* $\theta_1\in(\theta_{\max},\pi)$ *and then increases for* $\theta_1<\theta<\pi$.

Lemma 4 *For* $\rho>1$, *the condition* $\rho\cos\theta=-1$ *determines a unique angle* $\theta_{\min}\in(\pi/2,\pi)$ *for which...*

$$\psi : [-\theta_{\min},\theta_{\min}]\times(1,\infty)\to[-\pi,\pi],$$

and we use the same formula (10) obtaining a homotopy $H:[-\pi,\pi]\times(1,\infty)\to\mathbb{C}$.

Theorem 1 *Concatenating the homotopies* H *is Lemmas 1-4 we obtain an expanding radial loop with center* $P=0\in\mathbb{C}$.

The Final Case

For r very small then, we consider the expressions

$$\alpha(t) = q(re^{it}) + \frac{4b_0}{a_1^2} = (1 + |c_1|re^{it})^2 - 1 + \frac{4a_0}{a_1^2} \tag{12}$$

and

$$\alpha_0(t) = 2|c_1|re^{it} + \frac{4a_0}{a_1^2}. \tag{13}$$

The first expression (12) may be written as

$$\alpha(t) = 2|c_1|r\cos t + |c_1|^2r^2\cos(2t) + i[2|c_1|r\sin t + |c_1|^2r^2\sin(2t)] + \frac{4a_0}{a_1^2}.$$

The expression (13) is a parameterization of a circle with radius $2|c_1|r$ and center $4a_0/a_1^2 \in \mathbb{C}\backslash\{0\}$. We note also that

$$\begin{aligned}\alpha'(t) &= 2(1 + |c_1|re^{it})|c_1|r\ ie^{it} \\ &= -2|c_1|r\sin t - 2|c_1|^2r^2\sin(2t) + i[2|c_1|r\cos t + 2|c_1|^2r^2\cos(2t)] \\ &= 2|c_1|r\left(-\sin t - |c_1|r\sin(2t) + i[\cos t + |c_1|r\cos(2t)]\right),\end{aligned}$$

while

$$\alpha_0' = 2|c_1|r\left(-\sin t + i\cos t\right).$$

Using these calculations we estimate

$$|\alpha - \alpha_0| + |\alpha' - \alpha_0'| = |c_1|^2r^2 + 2|c_1|^2r^2 = 3|c_1|^2r^2.$$

We conclude that α converges to α_0 uniformly in $C^1[0, 2\pi]$ as a smooth parameterized simple closed curve as $r \searrow 0$. In particular, none of the small curves $p(\partial D_r(0))$ pass through $0 \in \mathbb{C}$.

We would like somewhat different information about the deforming curves $p(\partial D_r(0))$ applying to the entire interval $0 < r \leq 1/|c_1|$. Our objective is to express each of these curves in "polar coordinates" with respect to the center $4b_0/a_1^2$. To this end, we consider for $-\pi \leq t \leq \pi$ the expression

$$\alpha(t) - \frac{4a_0}{a_1^2} = (1 + |c_1|re^{it})^2 - 1.$$

For $0 \leq t \leq \pi$, we have

$$\mathrm{Re}[\alpha(t) - 4a_0/a_1^2] = 2|c_1|r\cos t + |c_1|^2r^2\cos(2t) = |c_1|r[2\cos t + |c_1|r\cos(2t)].$$

We can also write

$$\begin{aligned}2\cos t + |c_1|r\cos(2t) &= 2|c_1|r\cos^2 t + 2\cos t - |c_1|r \\ &= |c_1|r\left[2\left(\cos t + \frac{1}{2|c_1|r}\right)^2 - \frac{1}{2|c_1|^2r^2} - 1\right].\end{aligned}$$

Given that $\cos t$ decreases for $0 \le t \le \pi$ from $\cos 0 = 1$ to $\cos\pi = -1$ and $0 < 2|c_1|r < 2$ we know the value

$$\cos t + \frac{1}{2|c_1|r} \quad \text{decreases from} \quad 1 + \frac{1}{2|c_1|r} > \frac{1}{2} \quad \text{to} \quad -1 + \frac{1}{2|c_1|r} > -\frac{1}{2}$$

on the interval $0 \le t \le \pi$. There are two possibilites: If $r \le 1/(2|c_1|) = |a_1|/4$, then $2\cos t + |c_1|r\cos(2t)$ decreases from $|c_1|r + 2 > 2$ to $|c_1|r - 2 \le -3/2$ determining a unique value t_* with $0 < t_* < \pi$ and

$$2\cos t_* + |c_1|r\cos(2t_*) = 0.$$

If $r > 1/(2|c_1|) = |a_1|/4$, then there is a value t_0 with $\pi/2 < t_0 < \pi$ satisfying

$$\cos t_1 = -\frac{1}{2|c_1|r}$$

and for which the following holds:

$$\left(\cos t + \frac{1}{2|c_1|r}\right)^2 \quad \text{decreases from} \quad \left(1 + \frac{1}{2|c_1|r}\right)^2 > 0 \quad \text{to} \quad 0$$

on the interval $0 \le t < t_1$ and

$$\left(\cos t + \frac{1}{2|c_1|r}\right)^2 \quad \text{increases from} \quad 0 \quad \text{to} \quad \left(-1 + \frac{1}{2|c_1|r}\right)^2 > 0.$$

In this case $2\cos t + |c_1|r\cos(2t)$ decreases from $|c_1|r + 2 > 2$ to

$$2\cos t_1 + |c_1|r\cos(2t_1) = |c_1|r\left[-\frac{1}{2|c_1|^2r^2} - 1\right] = -\left(\frac{1}{2|c_1|r} + |c_1|r\right) < 0$$

on the interval $0 \le t < t_1$ determining a unique value t_* with $0 < t_* < t_1$ and

$$2\cos t_* + |c_1|r\cos(2t_*) = 0.$$

For $t_1 \le t \le \pi$

$|c_1|r - 2 \le -3/2$ determining a unique value t_* with $0 < t < \pi$ and

$$2\cos t_* + |c_1|r\cos(2t_*) = 0.$$

5 Summary

We have obtained five distinct cases/pictures.

5.1 Single Root at $0 \in \mathbb{C}$

5.2 Single Root in $\mathbb{C}\backslash\{0\}$

This case is represented by the single root of $w(\zeta) = (\zeta + 1)^2$ at $\zeta = -1$.

5.3 Distinct Roots $0 \in \mathbb{C}$ and $z \in \mathbb{C}\backslash\{0\}$

5.4 Distinct Roots $\pm z \in \mathbb{C}\backslash\{0\}$ (balanced)

5.5 Distinct Roots $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ with $z_2 \neq -z_1$ (nonzero and unbalanced)