Chapter 4

Chapter IV Complex integration

4.1 Lecture 9: § 1.1 Path integrals

Let $\gamma : [a,b] \to \mathbb{C}$, differentiable or piecewise differentiable, define a directed path in $\mathbb{C}$. Assume $f = u + iv : \{\gamma\} \to \mathbb{C}$ is continuous. Then we define

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\, \gamma'(t)\, dt \\ &= \int_a^b (ux' - vy')\, dt + i \int_a^b (uy' + vx')\, dt.\end{aligned}$$

In the definition above $x = \operatorname{Re} \gamma$ and $y = \operatorname{Im} \gamma$. Notice the following:

1. The function f need only be defined on $\{\gamma\}$; we can consider $\{\gamma\}$ as a (topological) subset of $\mathbb{C}$ for continuity.

2. In particular, we do not need f to be analytic, though we have used the same notation/convention $f = u + iv$.

3. This definition may be viewed as a special case of integrating $g : [a,b] \to \mathbb{C}$ according to

$$\int_a^b g(t)\, dt = \int_a^b \operatorname{Re} g(t)\, dt + i \int_a^b \operatorname{Im} g(t)\, dt.$$

4. The integral depends on the *direction* of the path. For example, if $\gamma(t) = t$ and $f(z) = z$, then

$$\int_\gamma f = \int_a^b t\,dt = \frac{1}{2}(b^2 - a^2).$$

But

$$\int_{-\gamma} f = \int_{-b}^{-a} (-t)(-1)\,dt = \frac{1}{2}(a^2 - b^2).$$

$(-\gamma : [-b, -a] \to \mathbb{C}$ by $-\gamma(t) = -t.)$

5. The integral does not depend on parameterization:

Proposition 3 *If $\phi : [\alpha, \beta] \to [a, b]$ is increasing (say differentiable with $\phi' > 0$ for simplicity) and onto, then*

$$\int_\gamma f = \int_\alpha^\beta f(\gamma \circ \phi(t))(\gamma \circ \phi)'(t)\,dt.$$

Proof: The usual change of variables applies in the real and imaginary parts:

$$\begin{aligned}\int_\gamma f &= \int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt \\ &= \int_\alpha^\beta [u(\gamma \circ \phi)x' \circ \phi - v(\gamma \circ \phi)y' \circ \phi]\phi'(\tau)\,d\tau \\ &\qquad + i\int_\alpha^\beta [u(\gamma \circ \phi)y' \circ \phi + v(\gamma \circ \phi)x' \circ \phi]\phi'(\tau)\,d\tau.\end{aligned}$$

On the other hand,

$$(\gamma \circ \phi)' = x' \circ \phi\phi' + iy' \circ \phi\phi' = \gamma'\phi'$$

so that

$$f(\gamma \circ \phi)(\gamma \circ \phi)' = (ux' - vy')\phi' + (uy' + vx')\phi'. \qquad \square$$

Put another way,

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)\,dt &= \int_\alpha^\beta f((\gamma \circ \phi)(\tau))\gamma'(\phi(\tau))\phi'(\tau)\,d\tau \\ &= \int_\alpha^\beta f((\gamma \circ \phi)(\tau))(\gamma \circ \phi)'(\tau)\,d\tau.\end{aligned}$$

Lemma 24 *Given* $g : [a, b] \to \mathbb{C}$,

$$\left| \int_a^b g(t)\, dt \right| \le \int_a^b |g(t)|\, dt. \tag{4.1}$$

Proof: If $\int_a^b g(t)\, dt = 0$, then (4.1) clearly holds. Assume $\int_a^b g(t)\, dt \neq 0$. Then

$$c = \int_a^b \overline{g(t)}\, dt \Bigg/ \left| \int_a^b g(t)\, dt \right|$$

is well-defined and nonzero so that we have

$$\left| \int_a^b g(t)\, dt \right|^2 = \left(\int_a^b g(t)\, dt \right) \left(\int_a^b \overline{g(t)}\, dt \right). \tag{4.2}$$

Thus,

$$\left| \int_a^b g(t)\, dt \right| = \int_a^b cg(t)\, dt.$$

In particular, $\int_a^b cg(t)\, dt \in \mathbb{R}$. Therefore,

$$\begin{aligned} \int_a^b cg(t)\, dt &= \int_a^b \mathrm{Re}[cg(t)]\, dt \\ &\le \int_a^b |cg(t)|\, dt \\ &= |c| \int_a^b |g(t)|\, dt. \end{aligned}$$

Finally, from (4.2)

$$\left| \int_a^b \overline{g(t)}\, dt \right| \left| \int_a^b g(t)\, dt \right| = \left| \int_a^b g(t)\, dt \right|^2,$$

so $|c| = 1$ and (4.1) holds. □

Corollary 4

$$\left| \int_\gamma f \right| \le \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt.$$

Parameterization by arclength

Lemma 25 *The values of the integrals*

$$\int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \qquad \text{and} \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt$$

are independent of the parameterization.

Proof: Consider the second integral. Given $\phi : [\alpha, \beta] \to [a, b]$ as above

$[u(\gamma\circ\phi)+iv(\gamma\circ\phi)]\,|(\gamma\circ\phi)'| = [u(\gamma\circ\phi)+iv(\gamma\circ\phi)]\,|\gamma'\circ\phi|\,|\phi'| = [u(\gamma\circ\phi)+iv(\gamma\circ\phi)]\,|\gamma'\circ\phi|\,\phi'$.

Thus, changing variables in $\int_a^b f(\gamma(t))|\gamma'(t)|\,dt$ we find

$$\int_\alpha^\beta f(\gamma\circ\phi)\,|(\gamma\circ\phi)'|\,d\tau = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

The first integral is a special case of the second. □

Notation: The integral $\int_a^b f(\gamma(t))|\gamma'(t)|\,dt$ is called **the integral of f along γ with respect to arclength** or **the integral over γ as a point set** and is denoted by

$$\int_{\{\gamma\}} f = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

The first integral in Lemma 25 is simply $\int_{\{\gamma\}} |f|$. Sometimes people also write

$$\int_{\{\gamma\}} f = \int_\gamma f\,ds \quad \text{or} \quad \int_\gamma f\,|dz|.$$

Note:

$$\int_{\{\gamma\}} f = \int_{\{-\gamma\}} f.$$

Partial integrals

The numbers

$$\int_a^b (ux' - vy')\,dt \qquad \text{and} \qquad \int_a^b (uy' + vx')\,dt$$

appearing in the definition of $\int_\gamma f$ are also independent of parameterization and are sometimes denoted by

$$\int_\gamma (u\,dx - v\,dy) \qquad \text{and} \qquad \int_\gamma (u\,dy + v\,dx)$$

so that

$$\int_\gamma f\,dx = \int_a^b f(\gamma(t))\,x'(t)\,dt = \frac{1}{2}\left(\int_\gamma f + \overline{\int_\gamma \bar{f}}\right) \tag{4.3}$$

and

$$\int_\gamma f\,dy = \int_a^b f(\gamma(t))\,y'(t)\,dt = \frac{1}{2i}\left(\int_\gamma f - \overline{\int_\gamma \bar{f}}\right). \tag{4.4}$$

Nexercise 48 *Show $\int_\gamma f\,dx$ and $\int_\gamma f\,dy$ are independent of parameterization.*

Nexercise 49 *Show that if we define $\int_\gamma f\,dx$ and $\int_\gamma f\,dy$ directly by (4.3) and (4.4), then*

$$\int_\gamma f = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).$$

Sometimes

$$\overline{\int_\gamma \bar{f}} \quad \text{is denoted by} \quad \int_\gamma f\overline{dz}.$$

Nexercise 50 *Express $\int_\gamma f\overline{dz}$ in terms of the directed path $\overline{\gamma} : [a, b] \to \mathbb{C}$.*

4.2 Lecture 10: Integral representation formulas § 1.3-3.1 (excerpts)

The following material is useful in a variety of contexts. There are significantly more general versions of most of the material we are going to present. We are partially motivated by trying to show, as quickly as possible, that a holomorphic (complex differentiable) function is analytic (represented by a power series).[1]

[1] I will try to keep the distinction throughout the discussion, but sometimes I may "slip up" and write "analytic" for "holomorphic."