Equivalently, the limit of the modulus of this complex number is zero. And since the modulus of a quotient of complex numbers is the quotient of the moduli of those numbers, we can take the conjugate of the numerator and denominator and get the same limit. That is,

$$\lim_{h-ik\to 0}\frac{1}{h+ik}\left\{u(x+h,-y-k)-u(x,-y)-\left(h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}\right)\right.$$
$$\left.-\,i\left[v(x+h,-y-k)-v(x,-y)-\left(h\frac{\partial v}{\partial x}-k\frac{\partial u}{\partial x}\right)\right]\right\}=0.$$

Noting that

$$(h+ik)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)=h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}+i\left(k\frac{\partial u}{\partial x}-h\frac{\partial v}{\partial x}\right),$$

we see the limit above is exactly the limit

$$\lim_{h-ik\to 0}\frac{1}{h+ik}\left\{g(z+h+ik)-g(z)-(h+ik)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)\right\}$$

involving the difference quotient for g. Taking the limit as $h-ik\to 0$ is the same as taking the limit as $(h,k)\to(0,0)$ or as $h+ik\to 0$. We conclude g' exists and

$$g'(z)=\overline{f'(\bar{z})}.$$

We have shown that when f is analytic, then g is also analytic on the appropriate domain. Dustin Smith rightly points out that this computation essentially completes the problem since

$$f(z)=\overline{g(\bar{z})}.$$

That is, when g is analytic, then f is analytic by the same argument.

2.3 Lecture 5: § 1.3-4 polynomials and rational functions

2.3.1 polynomials

A complex polynomial of **degree** n is a function of the form

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,\qquad a_n\neq 0.$$

The complex number a_0 is called the **leading coefficient**; the polynomial is called **monic** if $a_n = 1$.

Nexercise 13 *Show the polynomial above is analytic with*

$$P'(z) = \sum_{j=0}^{n} j a_j z^{j-1}.$$

If P and Q are polynomials, we say P is **divisible** by Q if there is some polynomial $q = q(z)$ such that $P = qQ$. In this case, we write

$$Q \big| P \qquad \text{"}Q \textbf{ divides } P\text{."}$$

Notice that if Q divides P, then $\deg(Q) \leq \deg(P)$. More generally, we have the **division algorithm** for polynomials:

Proposition 2 *If* $\deg(Q) \leq \deg(P)$ *and Q is monic, then there are unique polynomials $q = q(z)$ and $r = r(z)$ with* $\deg(r) < \deg(Q)$ *and*

$$P = qQ + r.$$

The polynomial q is called the **quotient** *and the polynomial r is called the* **remainder**.

Example 3 $P = z^4 + 4$; $Q = z^2 + 1$. $q = z^2 - 1$; $r = 5$.

$$z^4 + 4 = (z^2 - 1)(z^2 + 1) + 5.$$

Nexercise 14 *You should make sure that given any two polynomials P and Q with* $\deg(P) > \deg(Q)$, *you can find polynomials q and r with* $\deg(r) < \deg(Q)$ *and*

$$P = qQ + r.$$

Nexercise 15 *The division algorithm still holds if* $\deg(Q) > \deg(P)$. *(Explain.)*

Nexercise 16 *Prove the division algorithm by induction on the degree of P.*

roots

If $P(z_0) = 0$, then z_0 is called a **root** of P.

Theorem 6 *If $P(z_0) = 0$, then $(z - z_0)\big|P$.*

Proof: By the division algorithm

$$P = q(z - z_0) + r$$

where r is a constant. Since $P(z_0) = 0$, we know the constant $r = 0$. □

Theorem 7 (fundamental theorem of algebra) *We state the result in two equivalent ways:*

(version 1) *Given a polynomial P with degree n, there are n complex roots, $\alpha_1, \ldots, \alpha_n$ of P such that*

$$P(z) = a_n \prod_{j=1}^{n} (z - \alpha_j). \tag{2.4}$$

(version 2) *If P is a polynomial with $\deg(P) \geq 1$, then there is a complex number α such that $P(\alpha) = 0$, i.e., P has a (complex) root.*

multiplicity

The roots of a polynomial may not be all different from one another, i.e., *distinct,* i.e., there may be **repeated roots**. But the roots of a polynomial $\{\alpha_1, \ldots, \alpha_n\}$ are unique as a set, even counted with multiplicites. That is, up to ordering, the product in (2.4) is unique. In view of this situation, let us change notation slightly. Let the distinct roots be $\alpha_1, \ldots, \alpha_k$ with $k \leq n$, and let the root α_j have **multiplicity** m_j. Then the product expression for P becomes

$$P(z) = a_n \prod_{j=1}^{k} (z - \alpha_j)^{m_j}.$$

The power/multiplicity m_j is also called the **order of the zero** α_j, and the zero is said to be **simple** if $m_j = 1$.

Note that we have

$$\sum_{j=1}^{k} m_j = n.$$

The order of a zero m_j is related to differentiation of P. For example, if α_j is a simple zero, then

$$P(z) = (z - \alpha_j)Q(z) \qquad \text{with } Q(\alpha_j) \neq 0.$$

This means

$$P'(z) = Q(z) + (z - \alpha_j)Q'(z) \qquad \text{and} \qquad P'(\alpha)Q(\alpha_j) \neq 0.$$

More generally, we have the following result:

Lemma 6 *The zero α of a polynomial P has order m if and only if*

$$P^{(j)}(\alpha) = 0, \text{ for } j = 0, \ldots, m-1 \qquad \text{but} \qquad P^{(m)}(\alpha) \neq 0. \tag{2.5}$$

Proof: We first assume α has order m. Then $P = (z-\alpha)^m Q$ where Q is a polynomial with $Q(\alpha) \neq 0$. This means, for example,

$$P' = m(z-\alpha)^{m-1}Q + (z-\alpha)^m Q' \qquad \text{and} \qquad P'(\alpha) = m(\alpha-\alpha)^{m-1}Q(\alpha) = 0$$

unless $m = 1$. If $m = 1$, then the conclusion (2.5) of the theorem holds. If $m > 1$, we can continue and write

$$P' = (z-\alpha)^{m-1}[mQ + (z-\alpha)Q'] = (z-\alpha)^{m-1}Q_1$$

where Q_1 is a polynomial with $Q_1(\alpha) \neq 0$. Evidently, we can repeat this argument to find

$$P^{(j)} = (z-\alpha)^{m-j}Q_j \quad \text{with} \quad Q_j(\alpha) \neq 0 \quad \text{for} \quad j = 0, \ldots, m-1.$$

Differentiating once more, we obtain

$$P^{(m)} = Q_{m-1} + (z-\alpha)Q'_{m-1}.$$

Plugging in $z = \alpha$ into these relations, we get the conclusion (2.5) of the theorem.

Conversely, let us assume (2.5) holds, but the order of α is ℓ. By repeating the calculation above for the order, we obtain

$$P^{(j)}(\alpha) = 0, \; j = 0, \ldots, m-1 \qquad \text{but} \qquad P^{(m)}(\alpha) \neq 0.$$

Comparing this to (2.5) we see immediately that

$$\max\{k : P^j(\alpha) = 0, \text{ for } j = 0, \ldots, k\} = \ell - 1 = m - 1.$$

Thus, $m = \ell$ is the order of α and we are done. $\square$

Lucas' theorem

This result may be viewed as a kind of exercise in understanding the structure of polynomials and the properties of complex numbers.

Theorem 8 (Lucas' theorem) *Let z_0 and w be fixed (determining and open left half plane in $\mathbb{C}$). If P is a polynomial and* $\operatorname{Re}[(\alpha - z_0)\overline{iw}] > 0$ *for all roots α of P, i.e.,*

$$\operatorname{Im}[(\alpha - z_0)\bar{w}] > 0 \qquad \text{for all roots } \alpha \text{ of } P,$$

then

$$\operatorname{Im}[(\beta - z_0)\bar{w}] > 0 \qquad \text{for all roots } \beta \text{ of } P'.$$

Proof: We first assume α has order m, so we have the product representation

$$P(z) = a_n \prod_{j=1}^{k} (z - \alpha_j)^{m_j}.$$

Differentiating, we see

$$P' = a_n \sum_{\ell=1}^{k} m_\ell (z - \alpha_\ell)^{m_\ell - 1} \prod_{j \neq \ell} (z - \alpha_j)^{m_j}.$$

From this expression we see

$$\frac{P'}{P} = \sum_{\ell=1}^{k} \frac{m_\ell}{z - \alpha_\ell} = \sum_{\ell=1}^{k} m_\ell \frac{\overline{z - \alpha_\ell}}{|z - \alpha_\ell|^2}.$$

If β is any complex number in the complementary closed half space, then β is not a root of P, and the function P'/P is well-defined and finite valued at β and has value

$$\frac{P'(\beta)}{P(\beta)} = \sum_{\ell=1}^{k} m_\ell \frac{\overline{\beta - \alpha_\ell}}{|\beta - \alpha_\ell|^2}.$$

On the other hand, we also have

$$\operatorname{Im}[(\beta - z_0)\bar{w}] \le 0,$$

but for each ℓ

$$(\beta - \alpha_\ell)\bar{w} = (\beta - z_0)\bar{w} - (\alpha_\ell - z_0)\bar{w},$$

so

$$\mathrm{Im}[(\beta - \alpha_\ell)\bar{w}] = \mathrm{Im}[(\beta - z_0)\bar{w}] - \mathrm{Im}[(\alpha_\ell - z_0)\bar{w}] < 0$$

or

$$\mathrm{Im}\left[\overline{(\beta - \alpha_\ell)}\, w\right] > 0 \qquad \text{for} \quad \ell = 1, \ldots, k.$$

Therefore, $wP'(\beta)/P(\beta)$ satisfies

$$\begin{aligned}\mathrm{Im}\left[w\frac{P'(\beta)}{P(\beta)}\right] &= \mathrm{Im}\left[\sum_{\ell=1}^{k} m_\ell \frac{\overline{(\beta-\alpha_\ell)}\, w}{|\beta-\alpha_\ell|^2}\right] \\ &= \sum_{\ell=1}^{k} \frac{\mathrm{Im}\left[\overline{(\beta-\alpha_\ell)}\, w\right]}{|\beta-\alpha_\ell|^2} \\ &> 0.\end{aligned}$$

Therefore, $P'(\beta) \neq 0$, and any root of P' must actually lie in the same open half plane with the roots of P. □

This means that if one takes the collection of roots $\{\alpha_1, \ldots, \alpha_k\}$ of a polynomial P and constructs from them the minimal convex polygon K containing them, i.e., the convex hull of the roots, then the roots of P' also lie in K.

2.3.2 rational functions

A rational function is a function of the form

$$R(z) = \frac{P(z)}{Q(z)} \qquad \text{where } P \text{ and } Q \text{ are polynomials.}$$

We can, and do, assume P and Q have no common factors, i.e., no common zeros. If $\beta \in \mathbb{C}$ is a root of Q, then we say R has a **pole** at $z = \beta$, and we define

$$R(\beta) = \infty.$$

This makes perfectly good sense on the Riemann sphere. That is to say, more generally:

> *Rational functions are naturally considered with domain and range in the Riemann sphere.*

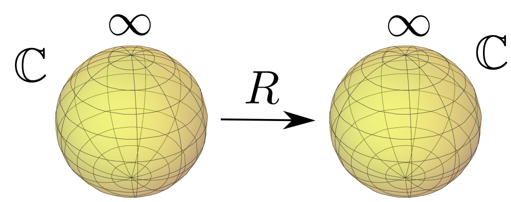

Figure 2.2: Can a rational function be defined on the entire Riemann sphere?

But there is a question: *Can we define $R(\infty)$?*

The answer is "yes," but it will take a little work and terminology to do so. Before we address the definition of $R(\infty)$ directly, let us focus on a finite pole.

First of all, the poles of R have an order, like the order of the zeros of P or of R. If β is a pole of R, then the **order of the pole** β is the order of β as a zero of Q. Let us consider the behavior of R in a neighborhood of a pole β of order m. The picture we should ultimately have in mind is that of the mapping $\tilde{R} = \sigma^{-1} \circ R \circ \sigma$. On the other hand, for the mapping R, we can write

$$R = \frac{P}{Q} = \frac{P}{(z-\beta)^m Q_1} = \frac{1}{(z-\beta)^m} \frac{P}{Q_1}$$

where Q_1 is a polynomial with a (finite) nonzero value at $z = \beta$. Thus,

$$R_1 = \frac{P}{Q_1}$$

is a rational function with a finite nonzero value $L \in \mathbb{C}$ at $z = \beta$. As $z \to \beta$, the factor $1/(z-\beta)^m$ satisfies

$$\left| \frac{1}{(z-\beta)^m} \right| = \frac{1}{|z-\beta|^m} \to +\infty.$$

This means for any $M > 0$, there is some $r > 0$ such that

$$|z-\beta| < r \qquad \text{implies} \qquad \left|\frac{P}{Q}\right| = \left|\frac{1}{(z-\beta)^m}\right| \left|\frac{P}{Q_1}\right| \geq \frac{2M}{|L|}\frac{|L|}{2} = M. \quad (2.6)$$

Put another way, there is some $\delta > 0$ and some $C > 0$ such that all points $z \in \mathbb{C}$ with $0 < d(z, \beta) < \delta$ satisfy

$$d(R(z), \infty) < C d(z, \beta)^m. \quad (2.7)$$

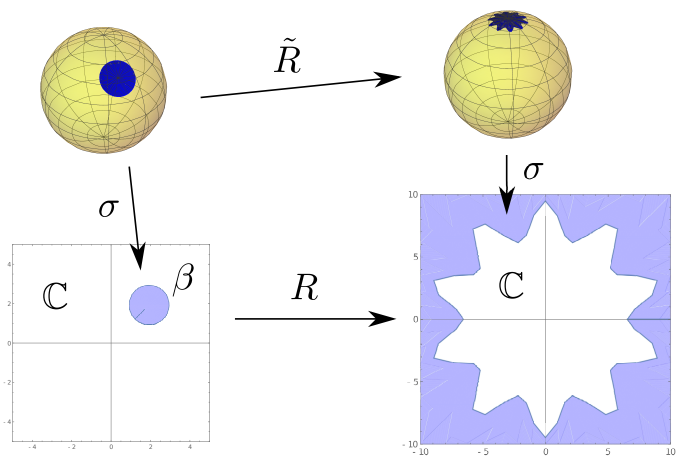

Figure 2.3: rational function at a finite pole

(Remember to think of these estimates/inequalities in the Riemann sphere.) We will give a rigorous proof of this fundamental observation about poles in a more general case below, but the basic assertion should be clear geometrically from (2.6) and what you "know" about stereographic projection.

Keep in mind that turning geometric ideas like this into explicit estimates is, more or less, what analysis is about.

While we are in the neighborhood of a finite pole, let us briefly consider the derivative of our rational function. The quotient rule applies, and we have

$$R' = \frac{QP' - PQ'}{Q^2} = \frac{(z-\beta)^m Q_1 P' - PQ'}{(z-\beta)^{2m} Q_1^2}.$$

On the other hand,

$$Q' = m(z-\beta)^{m-1}Q_1 + (z-\beta)^m Q_1' = (z-\beta)^{m-1}Q_{11}$$

where $Q_{11} = mQ_1 + (z-\beta)Q_1'$ is a polynomial with $Q_{11}(\beta) \neq 0$. Thus,

$$R' = \frac{(z-\beta)Q_1P' - PQ_{11}}{(z-\beta)^{m+1}Q_1^2}$$

has numerator $P_1 = (z-\beta)Q_1P' - PQ_{11}$ which is a polynomial with $P_1(\beta) = -P(\beta)Q_{11}(\beta) \neq 0$. We conclude that R' is a rational function with a pole of order $m+1$ at β.

Now, let us consider $R(\infty)$. In order to see what is happening, we make a change of variables and look at $R(1/\zeta)$ for ζ near $\zeta = 0$. We find

$$R\left(\frac{1}{\zeta}\right) = \frac{P(1/\zeta)}{Q(1/\zeta)}.$$

This is, of course, a rational function of ζ, and we claim the basic behavior of R at $z = \infty$ is determined, first of all, by the relative orders of P and Q. Let's say

$$P = \sum_{j=0}^{n} a_j z^j \qquad \text{and} \qquad Q = \sum_{j=0}^{m} b_j z^j \qquad \text{with} \qquad n > m.$$

Then

$$\begin{aligned} R\left(\frac{1}{\zeta}\right) &= \frac{a_n/\zeta^n + a_{n-1}/\zeta^{n-1} + \cdots + a_1/\zeta + a_0}{b_m/\zeta^m + b_{m-1}/\zeta^{m-1} + \cdots + b_1/\zeta + b_0} \\ &= \frac{a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n}{\zeta^{n-m}(b_0\zeta^m + b_1\zeta^{m-1} + \cdots + b_{m-1}\zeta + b_m)}. \end{aligned} \tag{2.8}$$

Since a_n and b_m are nonzero, $R(1/\zeta)$ has a pole of order $n-m$ at $\zeta = 0$. With this in mind let's think about what happens to points near the north pole under the mapping $\tilde{R} = \sigma^{-1} \circ R \circ \sigma$, but taking a detour via the reciprocal map $z \mapsto \zeta = 1/z$.

Nexercise 17 *What does the reciprocal map do to the Riemann sphere?*

Answer: It's a rotation by angle π about the x_1-axis. Points near the north pole correspond, under the reciprocal map to points near $\zeta = 0$. We know the images of these points tend to $w = \infty$ satisfying an estimate

$$|R(z)| \geq \frac{c}{|\zeta|^{n-m}} \qquad \text{as } \zeta \to 0 \tag{2.9}$$

where $c > 0$ is a constant. (Apply the estimate used to get (2.6) to (2.8) with $L = a_n/b_n$ and $c = |L|/2$.) Looking back at the spherical metric(s) (1.48)

$$d(z, w) = \cos^{-1}\left(1 - \frac{2|z-w|^2}{(|z|^2+1)(|w|^2+1)}\right); \qquad d(z,\infty) = \cos^{-1}\left(\frac{|z|^2-1}{|z|^2+1}\right),$$

$$\tilde{d}(z, w) = \frac{2|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}}; \qquad \tilde{d}(z,\infty) = \frac{2}{\sqrt{|z|^2+1}},$$

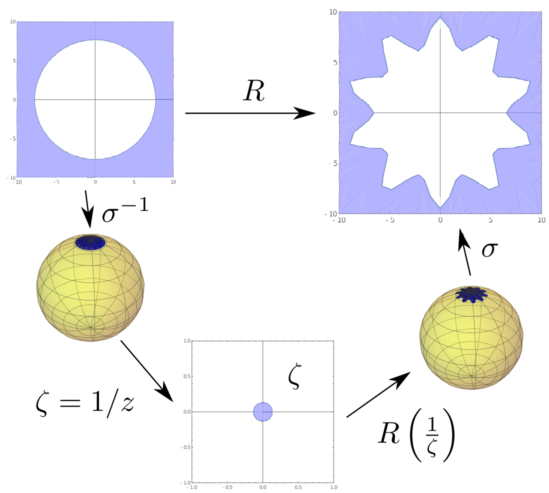

Figure 2.4: getting a look at infinity

we see

$$d(z,\infty) = d(\zeta,0) = \cos^{-1}\left(1 - \frac{2|\zeta|^2}{|\zeta|^2+1}\right) = \cos^{-1}\left(\frac{1-|\zeta|^2}{1+|\zeta|^2}\right)$$

where ∞ means the north pole $(0,0,1) \in \mathbb{S}^2$.

Nexercise 18 *There is a positive constant μ such that for ζ near $0 \in \mathbb{C}$,*

$$\frac{1}{\mu}\, d(\zeta,0) \leq |\zeta| \leq \mu\, d(\zeta,0). \tag{2.10}$$

Hint: $1 - x^2/2 \leq \cos x \leq 1 - x^2/4$ *and* $1 - 2|\zeta|^2 \leq (1-|\zeta|^2)/(1+|\zeta|^2) \leq 1 - |\zeta|^2$.

Solution: The cosine function has an alternating Taylor expansion $\cos x = \sum(-1)^j x^{2j}/(2j)!$. We can write this, on the one hand as

$$\cos x = 1 - \frac{x^2}{2} + \circ(x^4).$$

This means that when x is small, we can assume $\circ(x^4)/x^2 < 1/4$. Consequently, $\cos x \leq 1 - x^2/4$. This is one side of the first hint inequalities. Adding another couple terms,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \circ(x^6) = 1 - \frac{x^2}{2} + x^4\left(\frac{1}{24} + \frac{\circ(x^6)}{x^4}\right) \geq 1 - \frac{x^2}{2} \quad \text{for } x \text{ small.}$$

In fact, it is a nice fact to know (and a nice exercise to show) that $\cos x \geq 1 - x^2/2$ for all $x \in \mathbb{R}$. In any case, we have established the first hint inequalities. The second hint inequalities are almost immediate since

$$1 - \frac{2|\zeta|^2}{1+|\zeta|^2} = \frac{1-|\zeta|^2}{1+|\zeta|^2} \qquad \text{and} \qquad 1+|\zeta|^2 \geq 1.$$

Now, arc-cosine is a decreasing function on its domain $[-1, 1]$, so we can apply it to the first hint inequalities, as long as x is small and positive, to get

$$\cos^{-1}\left(1 - \frac{x^2}{4}\right) \leq x \leq \cos^{-1}\left(1 - \frac{x^2}{2}\right). \tag{2.11}$$

Similarly, applying arc-cosine to the second hint inequalities gives

$$\cos^{-1}\left(1 - |\zeta|^2\right) \leq d(\zeta, 0) = \cos^{-1}\left(\frac{1-|\zeta|^2}{1+|\zeta|^2}\right) \leq \cos^{-1}\left(1 - 2|\zeta|^2\right).$$

Writing $1 - |\zeta|^2 = 1 - x^2/2$ with $x = \sqrt{2}|\zeta|$, the right inequality of (2.11) implies

$$\sqrt{2}\,|\zeta| \leq d(\zeta, 0).$$

Similarly, writing $1 - 2|\zeta|^2 = 1 - x^2/4$ with $x = \sqrt{8}\,|\zeta|$, the left inequality of (2.11) implies

$$d(\zeta, 0)$$

We have shown

$$\sqrt{2}\,|\zeta| \leq d(\zeta, 0) \leq 2\sqrt{2}\,|\zeta|.$$

Since $1/(2\sqrt{2}) < \sqrt{2}$, we may take $\mu = 2\sqrt{2}$ and the assertion is established. □

Returning to (2.9) and applying the exercise, we have an estimate

$$|R(z)| \geq \frac{c/\mu^{n-m}}{d(z, \infty)^{n-m}}.$$

By a similar argument, this also implies that for some $M > 0$,

$$d(R(z), \infty) \leq M\, d(z, \infty)^{n-m}.$$

In fact, using the left side of (2.10), we see

$$d(R(z), \infty) = d\left(\frac{1}{|R(z)|}, 0\right) \leq \frac{\mu}{|R(z)|} \leq \frac{\mu}{c/\mu^{n-m}}\, d(z, \infty)^{n-m}.$$

In the situation we have described above, we say R has a pole of order $n - m$ at $z = \infty$. We can summarize our discussion as follows:

Definition-Proposition 3 *A rational function $R = P/Q$ has* **a pole at infinity of order** $\ell > 0$ *if any one of the following five equivalent conditions holds.*

1. *$R(1/\zeta)$ has a pole of order ℓ at $\zeta = 0$.*
2. $\deg(P) - \deg(Q) = \ell$.
3. *There is a constant $c > 0$ such that $(1/c)|z|^\ell \le |R(z)| \le c|z|^\ell$ for $|z|$ large.*
4. *There is a constant $c > 0$ such that*
$$\frac{1}{c\, d(z,\infty)^\ell} \le |R(z)| \le \frac{c}{d(z,\infty)^\ell} \qquad \textit{for } d(z,\infty) \textit{ small enough.}$$
5. *There is a constant $c > 0$ such that*
$$\frac{1}{c}\, d(z,\infty)^\ell \le d(R(z),\infty) c\, d(z,\infty)^\ell \qquad \textit{for } z \textit{ in some neighborhood of } \infty.$$

I think we have probably discussed the most difficult case. If $\deg(P) < \deg(Q)$ and we write $\deg(Q) - \deg(\ell) = \ell$, the expression in (2.8) becomes

$$R\left(\frac{1}{\zeta}\right) = \frac{\zeta^\ell(a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n)}{b_0\zeta^m + b_1\zeta^{m-1} + \cdots + b_{m-1}\zeta + b_m},$$

and we say R has a **zero of order** ℓ **at** $z = \infty$. Equivalently, the rational function $R(1/\zeta)$ has a zero of order ℓ at $\zeta = 0$, or, there are equivalent estimates

$$d(R(z),0) \sim d(z,\infty)^\ell \qquad \text{or} \qquad d(R(1/\zeta),0) \sim d(\zeta,0)^\ell$$

for $|z| = 1/|\zeta|$ large.

Nexercise 19 *Rigorously*[1] *justify the estimates given above for a rational function R with a zero of order ℓ at infinity where "$\sim$" means "is bounded below and above by constant multiples of."*

[1] If one wishes to avoid the estimates for cosine used with the intrinsic metric, one can obtain similar estimates for Ahlfors' extrinstic metric.

There is one final possibility: $\deg(P) = \deg(Q)$. In this case (2.8) becomes

$$R\left(\frac{1}{\zeta}\right) = \frac{a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n}{b_0\zeta^m + b_1\zeta^{m-1} + \cdots + b_{m-1}\zeta + b_m}, \qquad \text{with } a_n/b_n = L \in \mathbb{C}\backslash\{0\}.$$

Notice the left side $R(1/\zeta)$ is not (technically) defined for $\zeta = 0$, but the right side is well-defined for $\zeta = 0$. In such a case, we say this function of ζ has a **removable singularity**. (We shall see that this idea, and this term, may be applied more broadly to functions other than rational functions.)

On the other hand, with our new understanding of rational functions, we know the (rational) function $1/\zeta$ is defined at $\zeta = 0$ and takes the value ∞, so we can (and should) write

$$R(\infty) = L = a_n/b_n.$$

Nexercise 20 *Show that a rational function with a removable singularity at $z = \infty$ is continuous with respect to the metric on the Riemann sphere.*

Nexercise 21 *What can you say about the derivative of a rational function at a (finite) zero? What about at infinity?*

Definition-Proposition 4 *The* **order of a rational function** *is the total number of zeros (including those at ∞) counted with multiplicities. This is the same as the number of poles counted with multiplicities and is*

$$\max\{\deg(P), \deg(Q)\}.$$

Proof: Let $R = P/Q$ be a rational function with $\deg(P) = n < m = \deg(Q)$. Then R has $n = \deg(P)$ zeros and

$$R\left(\frac{1}{\zeta}\right) = \zeta^{m-n}\frac{\prod_{j=1}^{n}(1-\alpha_j\zeta)}{\prod_{j=1}^{m}(1-\beta_j\zeta)}$$

which has a zero of order $m - n$ at $\zeta = 0$, i.e., R has a zero of order $m - n$ at ∞. For the total number of zeros we have

$$n \text{ (finite zeros)} \ + \ (m-n) \text{ (at infinity)} \ = \ m \text{ total}.$$

Notice that R has m finite poles in this case, and no poles at ∞. The other cases are similar. $\square$

Theorem 9 *Every rational function of order n has (exactly) n zeros, n poles, and takes every value exactly n times (counting multiplicities in the extended complex plane).*

Proof: If $w \in \mathbb{C}$ (and R is a rational function), then

$$R(z) - w$$

is a rational function. The finite poles of $R(z) - w$ are the same as those of R. The poles of $R(z) - w$ at $z = \infty$ are the poles of $R(1/\zeta) - w$ at $\zeta = 0$, which are the same as the poles of R at ∞.

Thus, the total number of poles of $R(z)-w$, namely n, is the same as the total number of poles of R. This is the same as the total number of zeros of $R(z) - w$, and therefore $R(z)$ takes the value w a total number of n times. □

Metaprinciple of complex analysis 2 *One of the main objectives of complex analysis is to understand the values taken by complex analytic functions in various ways. The Riemann hypothesis is a question of this sort; one is interested in the location of the zeros of the Riemann zeta function.*

Metaprinciple of complex analysis 3 *If a function is defined for $z \in B_r(z_0)\backslash\{z_0\}$, then z_0 is called the* **puncture** *of the* **punctured neighborhood** *$B_r(z_0)\backslash\{z_0\}$, and we say f has an* **isolated singularity** *at z_0. Singularities can often be classified. The classifications of isolated singular points includes poles, removable singularites, and essential singularities.*

Metaprinciple of complex analysis 4 *If a function f has an isolated singularity at $z = \infty$, then one understands the behavior at the singularity by considering $g(\zeta) = f(1/\zeta)$ at $\zeta = 0$.*

2.3.3 linear fractional transformations

The function $f(z) = 1/z$ is a rational function of order 1. The rational functions of order 1 for a **group**; they are called linear fractional transformations (LFTs). Each has the form

$$R(z) = \frac{az+b}{cz+d}$$

for some complex numbers a, b, c, and d with $ad - bc \neq 0$. The inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{is} \qquad \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and the inverse of R is

$$R^{-1}(\zeta) = \frac{d\zeta - b}{-c\zeta + a}.$$

Nexercise 22 *What does $z \mapsto z + z_0$ do to the Riemann sphere? See the section on additional topics for this chapter.*

2.3.4 Expressing a rational function in terms of the singular behavior at the poles; partial fractions

Let $R(z) = P(z)/Q(z)$ be a rational fraction; as before P and Q are relatively prime polynomials, i.e., P and Q have no common factors. Let $\beta_1, \ldots, \beta_k$ be the finite poles of R, i.e., the zeros of Q. If R has no finite poles, then Q has degree zero and R is just a polynomial. We might as well assume this is not the case, so Q has some zeros.

Poles at infinity

It may be that R has also a pole at $z = \infty$ and this possibility is easy to to distinquish: If $\deg(P) \leq \deg(Q)$, then R remains bounded at $z = \infty$, and there is no pole at ∞. Otherwise, we can apply the division algorithm (perhaps partially) to write

$$R = zq_0(z) + c_0 + \frac{r_0(z)}{Q(z)} = zq_0(z) + \frac{c_0Q(z) + r_0(z)}{Q(z)}$$

where c_0 is a constant and $h_0(z) = c_0Q(z) + r_0(z)$ has degree less than or equal to that of Q.

The polynomial $h_0(z) = zq_0(z)$ is called the **singular part of R at infinity**, and the order of the pole at $z = \infty$ is $\deg(zq_0(z))$. Notice the constant c_0 is not singular, and we have not included it in the singular part of R at infinity.

Singular part at each finite pole

We now apply a similar process to isolate the singular behavior at each finite pole $\beta = \beta_j$. We first need a change of variables which moves the pole to the origin and then to ∞, namely

$$\zeta = \frac{1}{z-\beta}.$$

Thus, we consider

$$R\left(\frac{1}{\zeta} + \beta\right)$$

which has a pole at ∞ with the same singular behavior R has at β. For example,

$$R(z) = \frac{z^2+z-1}{z^2-1} = \frac{z^2+z-1}{(z+1)(z-1)}$$

has a simple pole at $\beta = -1$. To find the singular part at $\beta = -1$, we consider

$$R_\beta(\zeta) = R\left(\frac{1}{\zeta} - 1\right) = \frac{\left(\frac{1}{\zeta}-1\right))^2 + \frac{1}{\zeta} - 1 - 1}{\frac{1}{\zeta}\left(\frac{1}{\zeta}-2\right)} = \frac{1-\zeta-\zeta^2}{1-2\zeta}.$$

Dividing $\zeta^2+\zeta-1$ by $2\zeta-1$, we find the singular part of $R_\beta(\zeta)$ at $\zeta = \infty$:

$$R_\beta(\zeta) = \frac{1}{2}\zeta + \frac{\frac{3}{2}\zeta - 1}{2\zeta - 1}.$$

The singular part of R at $\beta = -1$ is now defined to be

$$\frac{1}{2}\left(\frac{1}{z-\beta}\right).$$

In general, the singular part of $R_\beta(\zeta) = R(\beta_j + 1/\zeta)$ at $\zeta = \infty$ will have the form $\zeta q_j(\zeta)$ for some polynomial q_j, and we define the **singular part of R at the finite pole** $z = \beta_j$ to be

$$G_j\left(\frac{1}{z-\beta_j}\right) = \frac{1}{z-\beta_j} q_j\left(\frac{1}{z-\beta_j}\right).$$

Finally, we consider

$$c(z) = R(z) - h_0(z) - \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right).$$

Lemma 7 *The function $c(z) = c$ is constant, so any rational function has a partial fractions decomposition*

$$R(z) = h_0(z) + c + \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right)$$

where $h_0(z) = zq_0(z)$ is the singular part of R at ∞ and

$$G_j\left(\frac{1}{z-\beta_j}\right) = \frac{1}{z-\beta_j} q_j\left(\frac{1}{z-\beta_j}\right)$$

is the singular part of R at the pole β_j.

Proof: The function $c(z)$ can have poles only (possibly) at $z = \infty$ or $z = \beta_j$ for some $k = 1, \ldots, k$. At $z = \infty$, we know

$$R(z) - h_0(z) = R(z) - zq_0(z) = c_0 + \frac{r_0(z)}{Q(z)}$$

is finite valued, so $c = c(z)$ has no pole at $z = \infty$.

Similarly,

$$R(z) - G_j\left(\frac{1}{z-\beta_j}\right) = c_j + \frac{q_0\left(\frac{1}{z-\beta_j}\right)}{Q\left(\frac{1}{z-\beta_j}\right)} \tag{2.12}$$

for some constant c_j and some polynomial q_j with $\deg(q_j) \leq \deg(Q)$. This means the right side of (2.12) is finite valued at $z = \beta_j$. Thus, $c = c(z)$ has no pole at $z = \beta_j$. Since $c = c(z)$ is a rational function with no pole, including no pole at ∞, it must be a constant. $\square$

Example 4

$$R(z) = \frac{z}{(z+1)(z-1)}.$$

There is no pole (and no singular part) at $z = \infty$. At $z = -1$, we consider

$$\frac{\frac{1}{\zeta} - 1}{\frac{1}{\zeta}\left(\frac{1}{\zeta} - 2\right)} = \frac{\zeta^2 - \zeta}{2\zeta - 1} = \frac{1}{2}\zeta - \frac{\frac{1}{2}\zeta}{2\zeta - 1}.$$

The singular part at $z = -1$ is

$$\frac{1}{2}\left(\frac{1}{\zeta+1}\right).$$

At $z = -1$,

$$\frac{\frac{1}{\zeta}+1}{\left(\frac{1}{\zeta}+2\right)\frac{1}{\zeta}} = \frac{\zeta^2+\zeta}{2\zeta+1} = \frac{1}{2}\zeta + \frac{\frac{1}{2}\zeta}{2\zeta+1}.$$

The singular part of R at $z = 1$ is

$$\frac{1}{2}\left(\frac{1}{\zeta-1}\right).$$

Subtracting the singular parts

$$\frac{z}{(z+1)(z-1)} - \frac{1}{2}\left(\frac{1}{\zeta+1}\right) - \frac{1}{2}\left(\frac{1}{\zeta-1}\right) = c.$$

Evaluating at $z = 0$, we find $c = 0$. Therefore,

$$\frac{z}{(z+1)(z-1)} = \frac{1}{2}\left(\frac{1}{\zeta+1}\right) + \frac{1}{2}\left(\frac{1}{\zeta-1}\right).$$

2.3.5 § 1.4

Exercise 1

$$R(z) = \frac{z^4}{z^3-1}.$$

At $z = \infty$,

$$R(z) = z + \frac{z}{z^3-1}.$$

At $z = 1$,

$$R(1/\zeta+1) = \frac{1/\zeta+1}{(1/\zeta+1)^3-1} = \frac{\zeta^3+\zeta^2}{(\zeta+1)^3-\zeta^3} = \frac{\zeta^3+\zeta^2}{3\zeta^2+3\zeta+1}$$

which has singular part $\zeta/3$ at ∞. Therefore, R has singular part at $z = 1$ given by

$$\frac{1}{3}\left(\frac{1}{z-1}\right).$$

At $z = (-1 \pm i\sqrt{3})/2$,

$$\begin{aligned}\frac{1/\zeta + (-1\pm i\sqrt{3})/2}{[1/\zeta + (-1\pm i\sqrt{3})/2]^3 - 1} &= \frac{(-1\pm i\sqrt{3})\zeta^3/2 + \zeta^2}{[(-1\pm i\sqrt{3})\zeta/2]^3 - \zeta^3} \\ &= \frac{(-1\pm i\sqrt{3})\zeta^3/2 + \zeta^2}{3[(-1\pm i\sqrt{3})/2]^2\zeta^2 + 3(-1\pm i\sqrt{3})\zeta/2 + 1}.\end{aligned}$$