Lang's Theorem

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Consider a holomorphic function $f : \Omega \to \mathbb{C}$ defined on an open set $\Omega \subset \mathbb{C}$.

Theorem 1 *If α_0 and α_1 are homotopic*[1] *in Ω, then*

$$\int_{\alpha_1} f = \int_{\alpha_0} f.$$

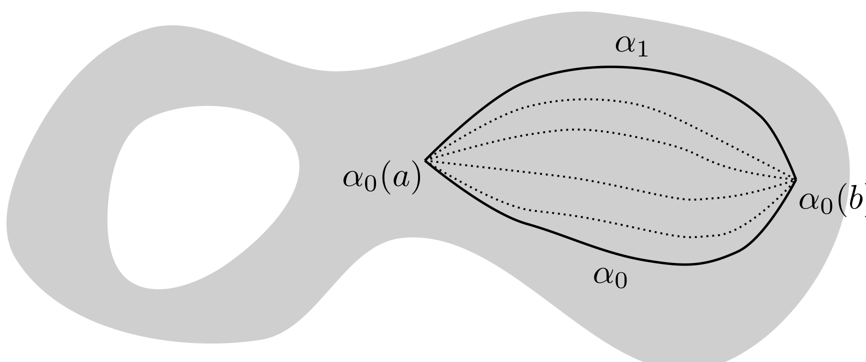

Figure 1: Homotopic curves within a domain of holomorphicity.

I first read about this result in Serge Lang's book on complex analysis. I went through (and reconstructed) the proof in detail feeling that Lang, as is his usual practice, had given a nice suggestion of how the proof went without quite giving "adequate" detail. He certainly didn't give all the details. I remember feeling that my proof was rather

[1]By homotopic here, we mean "fixed endpoint" homotopic in Ω; this will be reviewed/clarified below.

too complicated, partially simply because it was a kind of technically difficult result to prove. Now I see the result appears as Theorem 5.1 of Chapter 3 in Stein and Shakarchi's book on complex analysis. I also feel the proof given there is a little bit lacking, so I will try again to give a complete proof. Hopefully, I will be more satisfied than I was with my last attempt.

1 Set up

We have a homotopy $H : [a, b] \times [0, 1] \to \Omega$ where we assume both paths α_0 and α_1 have domain the interval $[a, b]$ and the following hold:

$$\begin{aligned} H(t, 0) &\equiv \alpha_0(t), \\ H(t, 1) &\equiv \alpha_1(t), \\ H(a, \tau) &\equiv \alpha_0(a) = \alpha_1(a), \text{ and} \\ H(b, \tau) &\equiv \alpha_0(b) = \alpha_1(b). \end{aligned}$$

We note also that

$$K = H\left([a, b] \times [0, 1]\right) = \{H(t, \tau) : a \le t \le b,\ 0 \le \tau \le 1\} \subset\subset \Omega$$

where in this case "$\subset\subset$" means "is a compact set compactly contained in" the open set Ω. Finally, the homotopy H is uniformly continuous on the compact set $K_0 = [a, b] \times [0, 1]$. These are the basic ingredients in the proof, and they are relatively simple. The rest involves, to a certain extent, technicalities. Note that

$$\text{dist}(K, \partial\Omega) > 0.$$

Thus, we can fix a positive number r with

$$r < \text{dist}(K, \partial\Omega).$$

We will "grid up" or partition the rectangle K_0 based on partitions

$$\begin{aligned} &a = a_0 < a_1 < a_2 < \cdots < a_n = b \text{ and} \\ &0 = \sigma_9 < \sigma_1 < \sigma_2 < \cdots < \sigma_m = 1 \end{aligned}$$

with

$$\begin{aligned} a_j - a_{j-1} &= \frac{b - a}{n} \text{ for } j = 1, 2, \ldots, n \text{ and} \\ \sigma_k - \sigma_{k-1} &= \frac{1}{m} \text{ for } k = 1, 2, \ldots, m. \end{aligned}$$

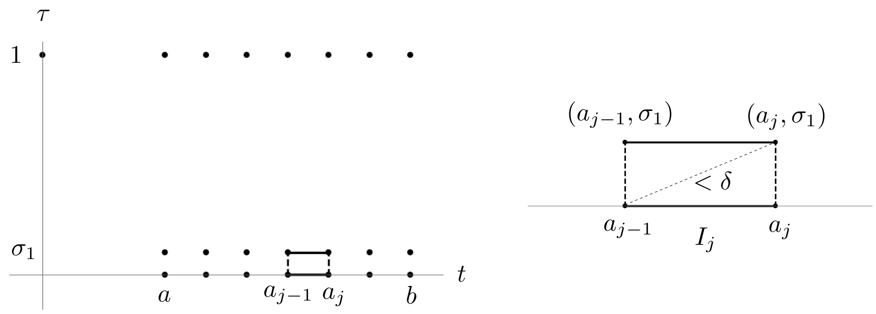

Figure 2: Partition of the rectangular domain of H (left) and a subrectangle along the bottom edge (right). The interval I_j is the domain of β_j and $\hat{\beta}_j$ which may be thought of as defined on the bottom and top of the subrectangle respectively.

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The basic idea is the following: For each $k = 0, 1, 2, \ldots, m$, we consider the path $\gamma_k : [a, b] \to \Omega$ by

$$\gamma_k(t) = H(t, \sigma_k).$$

Then we have $\gamma_0 = \alpha_0$ and $\gamma_m = \alpha_1$. Notice that each γ_k for $0 < k < m$ parameterizes a path connecting $\alpha_0(a)$ to $\alpha_0(b)$ in $K \subset \Omega$. We may consider the dashed curves in Figure 1 as illustrations of these paths. We will show (when we take the partition rectangles small enough) that

$$\int_{\alpha_0} f = \int_{\gamma_0} f = \int_{\gamma_1} f = \int_{\gamma_2} f = \cdots = \int_{\gamma_m} f = \int_{\alpha_1} f.$$

This will complete the proof.

2 Partition rectangles

By the uniform continuity of H, there is some $\delta > 0$ for which

$$|(t, \tau) - (t_0, \tau_0)| < \delta \qquad \Longrightarrow \qquad |H(t, \tau) - H(t_0, \tau_0)| < \frac{r}{2}$$

for any points (t, τ) and (t_0, τ_0) in K_0. With this in mind, we choose m and n large enough so that

$$\frac{b - a}{n} < \frac{\delta}{\sqrt{2}} \qquad \text{and} \qquad \frac{1}{m} < \frac{\delta}{\sqrt{2}}.$$

This has the consequence that each subrectangle $[a_{j-1}, a_j] \times [\sigma_{k-1}, \sigma_k]$ has diameter less than δ for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. Consequently, each of these subrectangles has image in the disk

$$D_{r/2}(p_{jk}) \qquad \text{where} \qquad p_{jk} = H(a_{j-1}, \sigma_{k-1})$$

is the image of the lower left corner.

This completes the basic set up. We need to use it to prove

$$\int_{\gamma_{k-1}} f = \int_{\gamma_k} f \qquad \text{for } k = 1, 2, \ldots, m.$$

3 Comparing integrals

This is where things get a bit complicated, and it's also where a main element of the proof arises. Let us set up that main element first:

Theorem 2 *(Cauchy's theorem in a disk) If $D_r(z_0) \subset\subset \Omega$, then f has a primitive in $D_r(z_0)$, that is, there is a holomorphic function $g : D_r(z_0) \to \mathbb{C}$ such that $g' = f$ on $D_r(z_0)$ and*

$$\int_\beta f = g(\beta(b)) - g(\beta(a))$$

for any path in $D_r(z_0)$ parameterized by β. In particular,

$$\int_\alpha f = 0$$

for any loop in $D_r(z_0)$ parameterized by α.

We apply this in disks with the radius $r < \text{dist}(K, \partial\Omega)$ mentioned above and centers determined by the partition values $a = a_0 < a_1 < a_2 < \cdots < a_n = b$. More precisely, let's start with the path $\alpha_0 : [a, b] \to \Omega$ for which

$$\alpha_0(t) = H(t, 0) = H(t, \sigma_0).$$

Let us consider α_0, furthermore, as a concatenation of paths $\beta_1, \beta_2, \ldots, \beta_n$ with

$$\beta_j : I_j = [a_{j-1}, a_j] \to \Omega \text{ by } \beta_j(t) = \alpha_0(t) = H(t, 0).$$

We consider also the "next" path $\gamma_1 : [a,b] \to \Omega$ by

$$\gamma_1(t) = H(t, \sigma_1)$$

as a concatenation of paths $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n$ with

$$\hat{\beta}_j : I_j = [a_{j-1}, a_j] \to \Omega \text{ by } \hat{\beta}_j(t) = \gamma_1(t) = H(t, \sigma_1).$$

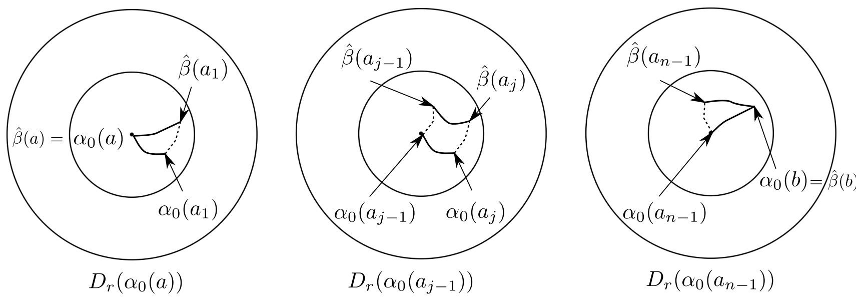

Figure 3: Image curves determined by a subrectangle on the bottom edge of the partiation. The image corresponding to the first subrectangle $[a, a_1] \times [0, \sigma_1]$ is illustrated on the left, and intermediate subrectangle as shown in Figure 2 in the middle, and the last subrectangle $[a_{n-1}, b] \times [0, \sigma_1]$ on the right. We have suppressed the index on $\hat{\beta} = \hat{\beta}_j$ in this illustration, taking $\hat{\beta}$ globally as another name for γ_1.

We begin by considering the lower left subrectangle $[a, a_1] \times [0, \sigma_1]$ whose image is illustrated on the left in Figure 3. Since this image lies entirely in $D_r(\alpha_0(a))$ where there is a primitive g_1 of f defined we can write

$$\begin{aligned} \int_{\hat{\beta}_1} f - \int_{\beta_1} f &= g_1(\hat{\beta}(a_1)) - g_1(\hat{\beta}(a)) - [g_1(\alpha_0(a_1)) - g_1(\alpha_1(a))] \\ &= g_1(\hat{\beta}(a_1)) - g_1(\alpha_0(a_1)) \end{aligned} \tag{1}$$

since at the left endpoints we have $H(a, \tau) \equiv \hat{\beta}a = \alpha_0(a)$.

We consider the next difference of integrals

$$\int_{\hat{\beta}_2} f - \int_{\beta_2} f$$

in reference to the middle illustration of Figure 3 where a primitive g_2 is defined on $D_r(\alpha_0(a_1))$. We find then

$$\int_{\hat{\beta}_2} f - \int_{\beta_2} f = g_2(\hat{\beta}_2(a_2)) - g_2(\hat{\beta}_2(a_1)) - [g_2(\alpha_0(a_2)) - g_2(\alpha_0(a_1))].$$

Since the primitives g_1 and g_2 are both defined on the intersection $D_r(\alpha_0(a)) \cap D_r(\alpha_0(a_1))$ contining both points $\hat{\beta}_1(a_1) = \hat{\beta}_1(a_1)$ and $\alpha_0(a_1)$ and satisfying there $g_2' = g_1' = f$, we conclude there is a constant $c_1 \in \mathbb{C}$ such that $g_2 = g_1 + c + 1$ on the common domain, and we find

$$\begin{aligned}\sum_{j=1}^{2} \int_{\hat{\beta}_j} f - \sum_{j=1}^{2} \int_{\beta_j} f = g_1(\hat{\beta}_1(a_1)) - g_1(\alpha_0(a_1)) + g_2(\hat{\beta}_2(a_2)) - g_1(\hat{\beta}_2(a_1)) - c_1 \\ - [g_2(\alpha_0(a_2)) - g_1(\alpha_0(a_1)) - c_1] \\ = g_2(\hat{\beta}_2(a_2)) - g_2(\alpha_0(a_2)).\end{aligned}$$

Taking the primitive g_3 on $D_r(\alpha_0(a_2))$ and repeating the same argument we obtian

$$\int_{\hat{\beta}_3} f - \int_{\beta_3} f = g_3(\hat{\beta}_3(a_3)) - g_3(\hat{\beta}_3(a_2)) - [g_3(\alpha_0(a_3)) - g_3(\alpha_0(a_2))].$$

Summing the differences again and taking account of the existence of a constant c_2 for which $g_3 = g_2 + c_2$, we have

$$\sum_{j=1}^{3} \int_{\hat{\beta}_j} f - \sum_{j=1}^{3} \int_{\beta_j} f = g_3(\hat{\beta}_3(a_3)) - g_3(\alpha_0(a_3)).$$

This kind of expression persists until

$$\sum_{j=1}^{n} \int_{\hat{\beta}_j} f - \sum_{j=1}^{n} \int_{\beta_j} f = g_n(\hat{\beta}_n(b)) - g_n(\alpha_0(b)). \tag{2}$$

At this point, we see as illustrated on the right in Figure 3 that $\hat{\beta}_n(b) = \alpha_0(b)$ so that the right side of (2) vanishes, and the left side is

$$\int_{\gamma_1} f - \int_{\alpha_0} f = 0.$$

Thus, we have completed the first step having verified

$$\int_{\alpha_0} f = \int_{\gamma_0} f = \int_{\gamma_1} f.$$

Each step showing

$$\int_{\gamma_k} f = \int_{\gamma_{k-1}} f \qquad \text{for} \qquad k = 2, 3, \ldots, n$$

is substantially the same as what we have done above leading inductively to the conclusion

$$\int_{\alpha_0} f = \int_{\gamma_n} f = \int_{\alpha_1} f$$

as asserted in Lang's theorem. Partially for the sake of completeness and partially as a review of the argument presented above, we include the details in the general case below.

4 The general step

Here we attempt to show

$$\int_{\gamma_k} f = \int_{\gamma_{k-1}} f.$$

The details differ from those of the first step above primarily in terms of notation and especially the subscripts associated with the concatenated curves β_j and $\hat{\beta}_j$ associated with γ_k and γ_{k+1} respectively. With appropriate changes of labeling, the illustrations of Figure 3 are still applicable. We have indicated the changes of labeling in Figure 4 with reference to the middle illustration of Figure 3. The modifications required to properly represent the beginning and ending illustrations on the left and right of Figure 3 in the general case are relatively easily obtained by drawing together/collapsing appropriate points.

We also emphasize below the transition involved in the addition/inclusion of the integrals

$$\int_{\beta_{j+1}} f \qquad \text{and} \qquad \int_{\hat{\beta}_{j+1}} f,$$

and with reference to this we have illustrated in Figure 4 the (four) disks of radius r and $r/2$ with centers at both $\beta_j(a_{j-1}) = \gamma_{k-1}(a_{j-1})$ and $\beta_{j+1}(a_j) = \gamma_{k-1}(a_j)$. The details are as follows:

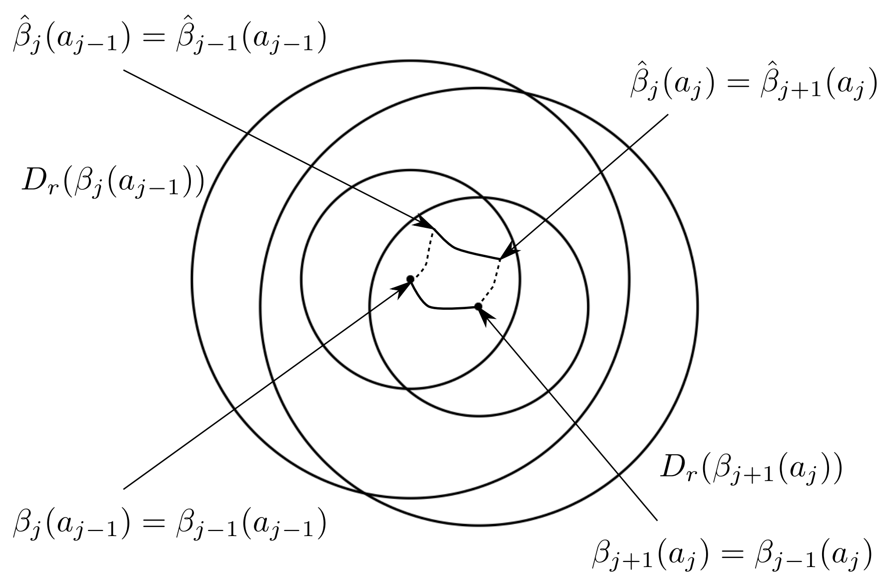

Figure 4: Image curves determined by a subrectangle on the bottom edge of the partiation.

Starting with the disk $D_r(\gamma_{k-1}(a)) = D_r(\beta_1(a))$ we have a primitive g_1 of f. Due to the uniform continuity assertion and the fineness of the rectangular grid the entire image of the subrectangle $[a, a_1] \times [\sigma_{k-1}, \sigma_k]$ under H lies in $D_{r/2}(\beta_1(a))$. We have then, precisely as in (1)

$$\begin{aligned}\int_{\hat\beta_1} f - \int_{\beta_1} f &= g_1(\hat\beta_1(a_1)) - g_1(\hat\beta_1(a)) - [g_1(\beta_1(a_1)) - g_1(\beta_1(a))] \\ &= g_1(\hat\beta_1(a_1)) - g_1(\beta_1(a_1)) \qquad (3)\end{aligned}$$

since $H(a,\tau) \equiv \hat\beta_1(a) = \beta_1(a)$.

We then consider the next difference of integrals

$$\int_{\hat\beta_2} f - \int_{\beta_2} f.$$

Much as before there exists a primitive g_2 on $D_r(\beta_2(a_1))$ so that

$$\int_{\hat\beta_2} f - \int_{\beta_2} f = g_2(\hat\beta_2(a_2)) - g_2(\hat\beta_2(a_1)) - [g_2(\beta_2(a_2)) - g_2(\beta_2(a_1))].$$

Noting that

$$\beta_2(a_1) = \beta_1(a_1) = \gamma_{k-1}(a_1) \qquad \text{and} \qquad \hat{\beta}_2(a_1) = \hat{\beta}_1(a_1) = \gamma_k(a_1)$$

the difference can be rewritten as

$$\int_{\hat{\beta}_2} f - \int_{\beta_2} f = g_2(\hat{\beta}_2(a_2)) - g_2(\hat{\beta}_1(a_1)) - [g_2(\beta_2(a_2)) - g_2(\beta_1(a_1))].$$

Since $\beta_1(a_1) = \gamma_{k-1}(a_1)$ is also the center of the (next) disk $D_r(\beta_2(a_1))$, the uniform continuity assertion and the fineness of the grid also implies the entire image of the first subrectangle $[a, a_1] \times [\sigma_{k-1}, \sigma_k]$ under H lies also in $D_{r/2}(\beta_2(a_1))$. In particular this image

$$\{H(t, \tau) : a \leq t \leq a_1,\ \sigma_{k-1} \leq \tau \leq \sigma_k\}$$

lies in the intersection $D_{r/2}(\beta_1(a)) \cap D_{r/2}(\beta_2(a_1))$ and also in the intersection $D_r(\beta_1(a)) \cap D_r(\beta_2(a_1))$ where both g_1 and g_2 are defined with $g_1' = g_2' = f$. We conclude there exists a constant $c_1 \in \mathbb{C}$ for which $g_2 = g_1 + c_1$ on this intersection. In particular, at the points $\beta_1(a_1) = \beta_2(a_1)$ and $\hat{\beta}_1(a_1) = \hat{\beta}_2(a_1)$ we can write

$$g_2(\beta_1(a_1)) = g_1(\beta_1(a_1)) + c_1 \qquad \text{and} \qquad g_2(\hat{\beta}_1(a_1)) = g_1(\hat{\beta}_1(a_1)) + c_1.$$

Substituting these values we find

$$\begin{aligned}\int_{\hat{\beta}_2} f - \int_{\beta_2} f &= g_2(\hat{\beta}_2(a_2)) - g_1(\hat{\beta}_1(a_1)) - c_1 \\ &\qquad - [g_2(\beta_2(a_2)) - g_1(\beta_1(a_1)) - c_1] \\ &= g_2(\hat{\beta}_2(a_2)) - g_2(\beta_2(a_2)) - [g_1(\hat{\beta}_1(a_1)) - g_1(\beta_1(a_1))].\end{aligned}$$

From this, we may conclude much as in the initial case

$$\sum_{j=1}^{2} \int_{\hat{\beta}_j} f - \sum_{j=1}^{2} \int_{\beta_j} f = g_2(\hat{\beta}_2(a_2)) - g_2(\beta_2(a_2)).$$

As before, the same approach applies to

$$\int_{\hat{\beta}_3} f - \int_{\beta_3} f = g_3(\hat{\beta}_3(a_3)) - g_3(\hat{\beta}_3(a_2)) - [g_3(\beta_3(a_3)) - g_3(\beta_3(a_2))]$$

using the primitive g_3 on $D_r(\beta_3(a_2)) = D_r(\beta_2(a_2))$. We conclude

$$\sum_{j=1}^{3} \int_{\hat{\beta}_j} f - \sum_{j=1}^{3} \int_{\beta_j} f = g_3(\hat{\beta}_3(a_3)) - g_3(\beta_3(a_3))$$

and

$$\sum_{j=1}^{\ell} \int_{\hat{\beta}_j} f - \sum_{j=1}^{\ell} \int_{\beta_j} f = g_\ell(\hat{\beta}_\ell(b)) - g_\ell(\beta_\ell(b)) \tag{4}$$

for $\ell = 2, 3, \ldots, n$. Taking (4) with $\ell = n$ gives

$$\int_{\gamma_k} f - \int_{\gamma_{k-1}} f = g_n(\gamma_k(b)) - g_n(\gamma_{k-1}(b)) = 0$$

since

$$H(b, \tau) \equiv \hat{\beta}_n(b) = \gamma_{k-1}(b) = \gamma_k(b) = \beta_n(b).$$