Notes on the Fourier Transform

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The Fourier transform is primarily a correspondence between pairs of functions of a **real variable**, though the values of the functions are naturally complex. More precisely, given a function $f \in C^0(\mathbb{R} \to \mathbb{C})$ satisfying, for some constant $A > 0$, a decay estimate

$$|f(x)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x \in \mathbb{R}, \tag{1}$$

we define the **Fourier transform** $\hat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i x \xi}\, dx. \tag{2}$$

Lemma 1 *Given the decay condition* (1)*, the value of the (hybrid) integral in* (2) *is a well-defined complex number, and the Fourier transform* $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$.

Proof: $\int_{-R}^{R} f(x)\ e^{-2\pi i x \xi}\, dx$ is well-defined as a hybrid integral simply because f is continuous. Also,

$$\begin{aligned}
\left| \int_{-R}^{R} f(x)\ e^{-2\pi i x \xi}\, dx \right| &\le \int_{-R}^{R} |f(x)|\, dx \\
&\le A \int_{-R}^{R} \frac{1}{1+x^2}\, dx \\
&= 2A \tan^{-1} R \qquad (3) \\
&\le A \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx \\
&= \pi A. \qquad (4)
\end{aligned}$$

In particular, $M = \int_{-\infty}^{\infty} |f(x)|\, dx \le \pi A < \infty$. Notice that for any $\epsilon > 0$, there is some $N > 0$ for which $R \ge N$ implies

$$\int_{-\infty}^{-R} |f(x)|\, dx + \int_{R}^{\infty} |f(x)|\, dx = M - \int_{-R}^{R} |f(x)|\, dx < \epsilon. \tag{5}$$

For $n = 1, 2, 3, \ldots$ let

$$I_n = \int_{-n}^{n} f(x) e^{-2\pi i x \xi}\, dx.$$

Then for $m > n$,

$$I_m - I_n = \int_{-m}^{-n} f(x) e^{-2\pi i x \xi}\, dx + \int_{n}^{m} f(x) e^{-2\pi i x \xi}\, dx,$$

and

$$|I_m - I_n| \le \int_{-m}^{-n} |f(x)|\, dx + \int_{n}^{m} |f(x)|\, dx \le \int_{-\infty}^{-n} |f(x)|\, dx + \int_{n}^{\infty} |f(x)|\, dx.$$

In view of the assertion associated with (5) it follows that $\{I_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, and there is a well-defined complex number

$$I = \lim_{n \nearrow \infty} I_n \in \mathbb{C}.$$

Also, in view of the assertion associated with (5) we can take for any $\epsilon > 0$ some $N > 0$ for which

$$\int_{-\infty}^{-R} |f(x)|\, dx + \int_{R}^{\infty} |f(x)|\, dx < \frac{\epsilon}{2}$$

whenever $R > N$. Therefore, if $R > N$ we can also take $n \in \mathbb{N}$ with $n > R$ and we will have

$$
\begin{aligned}
\left|\int_{-R}^{R} f(x)e^{-2\pi i x\xi}\,dx - I\right| &\le \left|\int_{-R}^{R} f(x)e^{-2\pi i x\xi}\,dx - I_n\right| + |I_n - I| \\
&= \left|\int_{-n}^{-R} f(x)e^{-2\pi i x\xi}\,dx + \int_{R}^{n} f(x)e^{-2\pi i x\xi}\,dx\right| \\
&\qquad + \left|\int_{-\infty}^{-n} f(x)e^{-2\pi i x\xi}\,dx + \int_{n}^{\infty} f(x)e^{-2\pi i x\xi}\,dx\right| \\
&\le \int_{-n}^{-R} |f(x)|\,dx + \int_{R}^{n} |f(x)|\,dx \\
&\qquad + \int_{-\infty}^{-n} |f(x)|\,dx \int_{n}^{\infty} |f(x)|\,dx \\
&\le \int_{-\infty}^{-R} |f(x)|\,dx + \int_{R}^{\infty} |f(x)|\,dx \\
&\qquad + \int_{-\infty}^{-R} |f(x)|\,dx \int_{R}^{\infty} |f(x)|\,dx \\
&< \epsilon.
\end{aligned}
$$

We have established then that

$$
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx = \lim_{R\nearrow\infty} \int_{-R}^{R} f(x)e^{-2\pi i x\xi}\,dx = I
$$

is a well-defined complex number (for each $\xi \in \mathbb{R}$). That is, $\hat{f} : \mathbb{R} \to \mathbb{C}$ is a well-defined function.

To see that $\hat{f}$ is continuous, we consider for $\xi_1, \xi_2 \in \mathbb{R}$ and $R > 0$

$$
\begin{aligned}
\hat{f}(\xi_2) - \hat{f}(\xi_1) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi_2}\,dx - \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi_1}\,dx \\
&= \int_{-\infty}^{-R} f(x)e^{-2\pi i x\xi_2}\,dx + \int_{R}^{\infty} e^{-2\pi i x\xi_2}\,dx \\
&\qquad - \int_{-\infty}^{-R} f(x)e^{-2\pi i x\xi_1}\,dx - \int_{R}^{\infty} e^{-2\pi i x\xi_1}\,dx \\
&\qquad + \int_{-R}^{R} f(x)\left(e^{-2\pi i x\xi_2} - e^{-2\pi i x\xi_1}\right)\,dx.
\end{aligned}
$$

Given $\epsilon > 0$, there is some $R > 0$ for which

$$\int_{-\infty}^{-R} |f(x)|\, dx + \int_{R}^{\infty} |f(x)|\, dx < \frac{\epsilon}{3}.$$

Having fixed R, there is some $\delta > 0$ such that for each x with $|x| \le R$ there holds

$$\left|e^{-2\pi i x\xi_2} - e^{-2\pi i x\xi_1}\right| = \left|e^{-2\pi i x(\xi_2-\xi_1)} - 1\right| < \frac{\epsilon}{3\left(1 + \int_{-\infty}^{\infty} |f(x)|\, dx\right)}.$$

Therefore, if $|\xi_2 - \xi_1| < \delta$

$$\begin{aligned}
\left|\hat{f}(\xi_2) - \hat{f}(\xi_1)\right| \le & \left|\int_{-\infty}^{-R} f(x)e^{-2\pi i x\xi_2}\, dx\right| + \left|\int_{R}^{\infty} f(x)e^{-2\pi i x\xi_2}\, dx\right| \\
& + \left|\int_{-\infty}^{-R} f(x)e^{-2\pi i x\xi_1}\, dx\right| + \left|\int_{R}^{\infty} f(x)e^{-2\pi i x\xi_1}\, dx\right| \\
& + \left|\int_{-R}^{R} f(x)\left(e^{-2\pi i x\xi_2} - e^{-2\pi i x\xi_1}\right)\, dx\right| \\
\le & \int_{-\infty}^{-R} |f(x)|\, dx + \int_{R}^{\infty} |f(x)|\, dx \\
& + \int_{-\infty}^{-R} |f(x)|\, dx + \int_{R}^{\infty} |f(x)|\, dx \\
& + \int_{-R}^{R} |f(x)|\left|e^{-2\pi i x(\xi_2-\xi_1)} - 1\right|\, dx \\
< & \ \epsilon.
\end{aligned}$$

This shows not only that $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$, but in fact, $\hat{f}$ is **uniformly continuous** on $\mathbb{R}$.[1] $\square$

Corollary 2 *Under the hypotheses of Lemma 1, namely that* $f \in C^0(\mathbb{R} \to \mathbb{C})$ *and for some* $A > 0$

$$|f(x)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x \in \mathbb{R},$$

we have that $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$ *given by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx \tag{6}$$

[1] This sounds somewhat impressive, but something rather stronger is true; see Exercise 2 below.

and $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

We claim the negative sign in the power of the exponential in the formula (6) has not been used in any essential way:

Lemma 3 *If $g \in C^0(\mathbb{R} \to \mathbb{C})$ satisfies for some $B > 0$ a decay estimate*

$$|g(\xi)| \le \frac{A}{1+\xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R},$$

we have that $\check{g} : \mathbb{R} \to \mathbb{C}$ defined by

$$\check{g}(x) = \int_{-\infty}^{\infty} g(\xi) e^{2\pi i \xi x}\, d\xi \tag{7}$$

1. *is well-defined,*
2. *satisfies $\check{g} \in C^0(\mathbb{R} \to \mathbb{C})$, and*
3. *also has $\check{g}$ is uniformly continuous on $\mathbb{R}$.*

While the formula (6) with the negative sign in the exponential gives the Fourier transform of f, the formula (7) with the positive sign is called the **Fourier inversion formula** or **inverse Fourier transform**. This formula is often written as

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, dx$$

under the assumption that a function $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$ is given—not necessarily the result of applying the Fourier transform to a function f, but simply a given function $g = \hat{f}$ satisfying the hypotheses on g in Lemma 3.

In order to apply these formulas successively, we need also to know that the resulting transformed function satisfies the required decay condition. We will obtain the required estimate for the Fourier transform under somewhat more restrictive conditions on the function f. More precisely, we will now consider restrictions of certain holomorphic functions. For $b > 0$, consider the strip

$$\Omega = \{z \in \mathbb{C} : |\operatorname{Im} z| < b\},$$

and consider holomorphic functions $f : \Omega \to \mathbb{C}$ satisfying for some $A > 0$ the uniform decay estimate

$$|f(x+iy)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x, y \in \mathbb{R} \quad \text{with} \quad x + iy \in \Omega. \tag{8}$$

This condition allows us to apply Lemma 1 to the quantity

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\, dx$$

with the values of the restriction

$$f\Big|_{\mathbb{R}}$$

taken in the place of those of a function $f \in C^0(\mathbb{R} \to \mathbb{C})$ as posited in the lemma. We can also use the values of f given by $g(x) = f(x + iy)$ along any other horizontal line $y = c$ for some $c \in \mathbb{R}$ with $|c| < b$. Then

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x + iy) e^{-2\pi i x\xi}\, dx$$

gives a well-defined uniformly continuous function.

Lemma 4 *(Theorem 2.1 in Stein and Shakarchi) For $f : \Omega \to \mathbb{C}$ holomorphic on the strip and satisfying the decay condition* (8) *as described above, we have for each fixed y with $0 \le y < b$*

$$|\hat{f}(\xi)| \le \pi A e^{-2\pi y|\xi|} \qquad \text{for} \qquad \xi \in \mathbb{R}. \tag{9}$$

It follows that for $y > 0$

$$|\hat{f}(\xi)| \le \frac{B}{1+\xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R} \tag{10}$$

where

$$B = \frac{\pi A}{\min\{1, 2\pi^2 y^2\}}.$$

Exercise 1 *Verify that for $y > 0$*

$$e^{-2\pi y|\xi|} < \frac{B}{1+\xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R} \tag{11}$$

with

$$B = B_0(y) = \frac{1}{\min\{1, 2\pi^2 y^2\}},$$

and find the smallest value of $B > 0$ for which (11) *holds. Note that*

$$\lim_{y \searrow 0} B_0(y) = +\infty,$$

Exercise 2 *Show that any continuous function $f \in C^0(\mathbb{R} \to \mathbb{C})$ satisfying the decay estimate (8) is uniformly continuous on all of $\mathbb{R}$. Also show that any function $g \in C^0(\mathbb{R} \to \mathbb{C})$ satisfying the decay estimate*

$$|g(\xi)| \le \pi A e^{-2\pi y|\xi|} \qquad \text{for} \qquad \xi \in \mathbb{R}$$

is uniformly continuous on all of $\mathbb{R}$. Give a general decay condition for a continuous function $f \in C^0(\mathbb{R} \to \mathbb{C})$ ensuring f is uniformly continuous on all of $\mathbb{R}$.

Proof of Lemma 4: Applying the estimate (4) we obtain the desired estimate (9) in the case $y = 0$.

For $0 < y < b$ and $\xi > 0$, we consider

$$\int_\alpha f(z)e^{-2\pi iz\xi}$$

with α a parameterization of $\partial\mathcal{R}$ where $\mathcal{R}$ is the rectangular domain

$$\mathcal{R} = (-R, R) \times (-y, 0).$$

This path consists of four segments which we parameterize by α_j for $j = 1, 2, 3, 4$ numbered counterclockwise starting with the bottom edge

$$\alpha_1(t) = t - yi \qquad \text{for} \qquad -R \le t \le R.$$

By Cauchy's theorem, since $f(z)e^{-2\pi iz\xi}$ is entire, we have

$$\sum_{j=1}^{4} \int_{\alpha_j} f(z)e^{-2\pi iz\xi} = \int_\alpha f(z)e^{-2\pi iz\xi} = 0. \tag{12}$$

Taking each portion of the integral in turn,

$$\begin{aligned} \int_{\alpha_1} f(z)e^{-2\pi iz\xi} &= \int_{-R}^{R} f(t - yi)e^{-2\pi i(t-yi)\xi}\, dt \\ &= \int_{-R}^{R} f(t - yi)e^{-2\pi it\xi}\, dt\ e^{-2\pi y\xi}. \end{aligned}$$

As mentioned above, the assertion of Lemma 1 applies to the function $g(t) = f(t - yi)$ in this case as well as the basic estimate (4) to give

$$\lim_{R \nearrow \infty} \int_{\alpha_1} f(z)e^{-2\pi iz\xi} = \hat{g}(\xi)\ e^{-2\pi y\xi}$$

with $|\hat{g}(\xi)| \leq \pi A$.

$$\int_{\alpha_2} f(z)e^{-2\pi iz\xi} = \int_{-y}^{0} f(R+ti)e^{-2\pi i(R+ti)\xi}\ (i)\, dt$$
$$= ie^{-2\pi iR\xi} \int_{-y}^{0} f(R+ti)e^{2\pi t\xi}\, dt.$$

Therefore,

$$\left|\int_{\alpha_2} f(z)e^{-2\pi iz\xi}\right| = \left|\int_{-y}^{0} f(R+ti)e^{2\pi t\xi}\, dt\right|$$
$$\leq \int_{-y}^{0} |f(R+ti)|e^{2\pi t\xi}\, dt$$
$$\leq \int_{-y}^{0} |f(R+ti)|\, dt$$
$$\leq \int_{-y}^{0} \frac{A}{1+R^2}\, dt$$
$$= \frac{Ay}{1+R^2},$$

and

$$\lim_{R\nearrow\infty} \left|\int_{\alpha_2} f(z)e^{-2\pi iz\xi}\right| = 0.$$

$$\lim_{R\nearrow\infty} \int_{\alpha_3} f(z)e^{-2\pi iz\xi} = -\lim_{R\nearrow\infty} \int_{-R}^{R} f(x)e^{-2\pi ix\xi}\, dx = -\hat{f}(\xi).$$

Finally,

$$\int_{\alpha_4} f(z)e^{-2\pi iz\xi} = -\int_{-y}^{0} f(-R+ti)e^{-2\pi i(-R+ti)\xi}\ (i)\, dt$$
$$= -ie^{2\pi iR\xi} \int_{-y}^{0} f(-R+ti)e^{2\pi t\xi}\, dt$$

so that

$$\begin{aligned}\left|\int_{\alpha_4} f(z)e^{-2\pi iz\xi}\right| &= \left|\int_{-y}^{0} f(-R+ti)e^{2\pi t\xi}\,dt\right| \\ &\le \int_{-y}^{0} |f(-R+ti)|e^{2\pi t\xi}\,dt \\ &\le \int_{-y}^{0} |f(-R+ti)|\,dt \\ &\le \int_{-y}^{0} \frac{A}{1+R^2}\,dt \\ &= \frac{Ay}{1+R^2},\end{aligned}$$

and

$$\lim_{R\nearrow\infty}\left|\int_{\alpha_4} f(z)e^{-2\pi iz\xi}\right| = 0.$$

Substituting these calculations in (12) in the form

$$-\int_{\alpha_3} f(z)e^{-2\pi iz\xi} = \int_{\alpha_1} f(z)e^{-2\pi iz\xi} + \int_{\alpha_2} f(z)e^{-2\pi iz\xi} + \int_{\alpha_4} f(z)e^{-2\pi iz\xi}$$

we find

$$\begin{aligned}\int_{-R}^{R}\int_{-R}^{R} f(x)e^{-2\pi ix\xi}\,dx = \int_{-R}^{R} & f(t-yi)e^{-2\pi it\xi}\,dt\ e^{-2\pi y\xi} \\ &+ ie^{-2\pi iR\xi}\int_{-y}^{0} f(R+ti)e^{2\pi t\xi}\,dt \\ &\quad - ie^{2\pi iR\xi}\int_{-y}^{0} f(-R+ti)e^{2\pi t\xi}\,dt,\end{aligned}$$

and taking the limit as $R\nearrow\infty$

$$\hat{f}(\xi) = \hat{g}(\xi)\ e^{-2\pi y\xi}.$$

In particular,

$$|\hat{f}(\xi)| \le \pi A\ e^{-2\pi y\xi}$$

as asserted in the statement of the lemma.

Exercise 3 *Show the same decay estimate* (9) *holds when* $\xi < 0$ *by computing*

$$\lim_{R \nearrow \infty} \int_\alpha f(z) e^{-2\pi i z \xi}$$

where α *parameterizes the boundary of the rectangular domain* $\mathcal{R} = (-R, R) \times (0, y)$.

This exercise completes the proof of Lemma 4. $\square$

Theorem 1 *(Fourier inversion formula; Theorem 2.2 in Stein and Shakarchi) For* $f : \Omega \to \mathbb{C}$ *holomorphic on the strip*

$$\Omega = \{x + iy : x \in \mathbb{R} \text{ and } |y| < b\}$$

and satisfying the decay condition (8) *namely for some fixed* $A > 0$

$$|f(x + iy)| \le \frac{A}{1 + x^2} \qquad \text{for} \qquad x, y \in \mathbb{R} \quad \text{with} \quad x + iy \in \Omega$$

so that $\hat{f} : \mathbb{R} \to \mathbb{C}$ *is well-defined by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\; e^{-2\pi i x \xi}\, dx,$$

we have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\; e^{2\pi i \xi x}\, d\xi.$$

1 Paley-Wiener Theorem(s)

In this section we discuss Theorem 3.3 of Stein and Shakarchi. The first result we state may be considered a modified version of Lemma 4 (Stein and Shakarchi's Theorem 2.1):

Theorem 2 *(First Paley-Wiener Theorem) Assume* $f : \mathbb{C} \to \mathbb{C}$ *is entire and satisfies the following decay conditions:*

(i) *There are some constants* $C > 0$ *and* $M > 0$ *for which*

$$|f(z)| \le C\; e^{2\pi M|z|} \qquad \text{for} \qquad z \in \mathbb{C}. \tag{13}$$

(ii) *There is some constant A for which*

$$|f(x)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x \in \mathbb{R}. \tag{14}$$

Then in addition to the basic assertions of Lemma 1 and Lemma 4 that $\hat{f} : \mathbb{R} \to \mathbb{C}$ exists, satisfies $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$, and us uniformly continuous on $\mathbb{R}$, there holds

$$\hat{f}(\xi) \equiv 0 \qquad \text{for} \qquad |\xi| \ge M.$$

That is, $\operatorname{supp} \hat{f} \subset [-M, M]$.

Note that the requirement that f is entire and that condition (13) holds globally on the complex plane are nominally additions to the hypotheses of Lemma 4. On the other hand, the strong decay assumption of (8) is only required to hold along the real axis, so that is much weaker than the decay hypothesis required in Lemma 4.

We state now also a converse:

Theorem 3 *(Second Paley-Wiener Theorem) Assume $f \in C^0(\mathbb{R} \to \mathbb{C})$ satisfies for some $A > 0$ the decay condition*

$$|f(x)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x \in \mathbb{R}.$$

Assume also that $\operatorname{supp} \hat{f} \subset [-M, M]$. *Then there exists a holomorphic extension of f to the entire complex plane and this extension $f : \mathbb{C} \to \mathbb{C}$ satisfies*

$$|f(z)| \le C\, e^{2\pi M|z|} \qquad \text{for} \qquad z \in \mathbb{C}.$$