Assignment 9: Meromorphic Functions and residue calculus
Due Tuesday April 12, 2022

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Problem 1 *(Casorati-Weierstrass example) Let* $w \in \mathbb{C}$. *Find a sequence of points* $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}\backslash\{0\}$ *with*

$$\lim_{n\to\infty} z_n = 0 \qquad \text{and} \qquad \left|e^{1/z_n} - w\right| < \frac{1}{n}.$$

Hint: Try to prove something much stronger.

Problem 2 *(stereographic projection; Ahlfors Chapter 1 §2.4) Let* $\sigma : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ *be stereographic projection from the sphere*

$$\mathbb{S}^2 = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi^2 + \eta^2 + \zeta^2 = 1\}$$

to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ *by*

$$\sigma(\xi, \eta, \zeta) = \begin{cases} \frac{1}{1-\zeta}(\xi + i\eta), & \zeta \neq 1 \\ \infty, & \zeta = 1. \end{cases}$$

(a) *Show*

$$\sigma(p)\overline{\sigma(-p)} = -1$$

for $p \in \mathbb{S}^2 \backslash \{(0,0,1)\}$.

(b) *If* $C = \partial D_r(z_0)$, *then what is*

$$\sigma^{-1}(C) = \{\sigma^{-1}(z) : z \in C\}?$$

(c) *What are the holomorphic functions* $f : \mathbb{C} \to \mathbb{C}$ *which extend to be meromorphic on the Riemann sphere?*

(d) *Given an open subset* Ω *of* $\mathbb{C}$, *is it always true that*

$$\sigma^{-1}(\Omega) \cup \{(0,0,1)\} = \sigma^{-1}(\Omega \cup \{\infty\})?$$

(e) *Find the rational functions on the Riemann sphere without any poles.*

Problem 3 *(removable singularities) Let* $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ *be holomorphic with an isolated singularity at* $z_0 \in \Omega$.

(a) *If* f *is bounded, show*

$$\lim_{\epsilon \searrow 0} \int_{\zeta = \beta} \frac{f(\zeta)}{\zeta - z_0} = 0$$

where $\beta = \beta_\epsilon$ *parameterizes* $\partial D_\epsilon(z_0)$.

(b) *If*

$$\lim_{z \to z_0} |(z - z_0) f(z)| = 0,$$

show

$$\lim_{\epsilon \searrow 0} \int_{\zeta = \beta} \frac{f(\zeta)}{\zeta - z_0} = 0$$

where $\beta = \beta_\epsilon$ *parameterizes* $\partial D_\epsilon(z_0)$.

(c) *Prove Ahlfors' theorem on removable singularities: If*

$$\lim_{z\to z_0} |(z-z_0)f(z)| = 0,$$

then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ *with*

$$g\big|_{\Omega\backslash\{z_0\}} \equiv f.$$

Hint: You should use analytic continuation (carefully).

Problem 4 *(meromorphic function on the Riemann sphere) Consider the rational function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *by*

$$f(z) = \frac{p(z)}{q(z)}$$

where

$$p(z) = \sum_{n=0}^{k} a_n z^n = a_k \prod_{j=1}^{k}(z - z_j) \qquad \text{and} \qquad q(z) = \sum_{n=0}^{\ell} b_n z^n = b_\ell \prod_{j=1}^{\ell}(z - w_j)$$

are polynomials with no common factors. Notice we are considering $f : \mathbb{C} \to \mathbb{C}$ *as meromorphic with* $f(w_j) = \infty$ *for each* $j = 1, 2, \dots, \ell$.

(a) *Show*

$$\lim_{z\to w_j} f(z) = \infty \qquad \text{for each } j = 1, 2, \dots, \ell.$$

(b) *What is the order of each pole* w_j*; how is it determined?*

(c) *Write* $\phi(\zeta) = f(1/\zeta)$ *as a rational function (ratio of polynomials) and consider* $\phi : \mathbb{C}\backslash\{0\} \to \mathbb{C} \cup \{\infty\}$ *as a meromorphic function with an isolated singularity at* $\zeta = 0$*. Classify the singularity of* ϕ.

(d) *Consider* $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ *and describe/classify the (local) behavior at* $z_0 = \infty$.

Problem 5 *(stereographic projection; Ahlfors Chapter 1 §2.4) Let* $\sigma : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ *be stereographic projection from the sphere*

$$\mathbb{S}^2 = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi^2 + \eta^2 + \zeta^2 = 1\}$$

to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ *by*

$$\sigma(\xi, \eta, \zeta) = \begin{cases} \frac{1}{1-\zeta}(\xi + i\eta), & \zeta \neq 1 \\ \infty, & \zeta = 1. \end{cases}$$

Find the formula for $\sigma^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{(0,0,1)\}$.

Problem 6 *(analytic continuation) Let* $f : \Omega \to \mathbb{C}$ *be a non-constant holomorphic function with an isolated zero at* $z_0 \in \Omega$.

(a) *Show there exists some* $r > 0$, *some* $k \in \mathbb{N}$, *and a non-vanishing holomorphic function* $g : D_r(z_0) \to \mathbb{C} \setminus \{0\}$ *for which*

$$f\Big|_{D_r(z_0)} \equiv (z - z_0)^k g.$$

Hint: Power series.

(b) *Show there exists a (unique) holomorphic function* $h : \Omega \to \mathbb{C}$ *for which*

$$f \equiv (z - z_0)^k h.$$

Can you say h *is non-vanishing? If not, give an example.*

(c) *Compute the* **logarithmic derivative**

$$\frac{f'}{f}$$

in terms of h. *What can you say about* $\lambda(z) = f'/f$, *for example, locally at* z_0*?*

Problem 7 *(Rouche's theorem; argument principle) Recall McCuan's version of Rouche's theorem: Let $f, g : \Omega \to \mathbb{C}$ be holomorphic and assume α parameterizes a simple loop in Ω for which there holds*

$$f(\alpha) \neq 0 \qquad \text{and} \qquad \left| \frac{g(\alpha)}{f(\alpha)} - 1 \right| < 1. \tag{1}$$

Then $g(\alpha) \neq 0$, and the number of zeros of g (counted with multiplicities) circumnavigated by α is the same as the number of zeros of f (counted with multiplicities) circumnavigated by α:

$$\int_\alpha \frac{g'}{g} = \int_\alpha \frac{f'}{f}.$$

Using α in the argument of a function here, e.g., $f(\alpha) \neq 0$, means (naturally) that the condition holds on the curve, e.g., f does not vanish at any point on the curve parameterized by α. Note that (1) implies $g(\alpha) \neq 0$.

(a) *Prove that McCuan's version implies Ahlfors' version: Let $f, g : \Omega \to \mathbb{C}$ be holomorphic and assume α parameterizes a simple loop in Ω for which there holds*

$$|f(\alpha) - g(\alpha)| < |f(\alpha)|. \tag{2}$$

Then $f(\alpha) \neq 0$, $g(\alpha) \neq 0$, and the number of zeros of g (counted with multiplicities) circumnavigated by α is the same as the number of zeros of f (counted with multiplicities) circumnavigated by α:

$$\int_\alpha \frac{g'}{g} = \int_\alpha \frac{f'}{f}.$$

(b) *Prove that McCuan's version implies Stein's version: Let $f, h : \Omega \to \mathbb{C}$ be holomorphic and assume α parameterizes a simple loop in Ω for which there holds*

$$|h(\alpha)| < |f(\alpha)|. \tag{3}$$

Then $f(\alpha) + h(\alpha) \neq 0$, and the number of zeros of $f + h$ (counted with multiplicities) circumnavigated by α is the same as the number of zeros of f (counted with multiplicities) circumnavigated by α:

$$\int_\alpha \frac{f'}{f} = \int_\alpha \frac{f' + h'}{f + h}.$$

Problem 8 *(S&S Chapter 3 Exercise 15(a)) Consider an entire function* $f : \mathbb{C} \to \mathbb{C}$ *for which there exists some* $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ *and positive constants* A *and* B *such that*

$$|f(z)| \leq A|z|^k + B.$$

Prove f *is a polynomial of degree no greater than* k.

Problem 9 *(S&S Chapter 3 Exercise 15(b)) Assume* $f : D_1(0) \to \mathbb{C}$ *is holomorphic and there exist arguments* θ_1 *and* θ_2 *with* $0 \leq \theta_1 < \theta_2 \leq 2\pi$ *such that* f *satisfies*

$$\lim_{r \to 1} f(re^{i\theta}) = 0 \qquad \textbf{uniformly } \textit{for } \theta_1 < \theta < \theta_2,$$

i.e., given any $\epsilon > 0$*, there exists some* δ *such that*

$$|f(re^{i\theta})| < \epsilon \qquad \textit{for } 1 - \delta < r < 1 \textit{ and } \theta_1 < \theta < \theta_2.$$

Show $f \equiv 0$.