Assignment 7: Analyticity
Due Tuesday March 29, 2022

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Problem 1 *(Dan Romik's notes: The Fundamental Theorem of Algebra) Recall that we were considering a monic quadratic polynomial* $p(z) = a_0 + a_1 z + z^2$ *under the assumptions*

(i) $a_0 \neq 0$,

(ii) $a_1 \neq 0$, *and*

(iii) $a_1^2 \neq 4a_0$.

In the previous assignment we considered

$$\{p(re^{it}) : 0 \le t \le 2\pi\}$$

for $r = \epsilon$ *small. Here we consider the opposite extreme*

$$\Gamma_\infty = \{p(Re^{it}) : 0 \le t \le 2\pi\}$$

for $R > 0$ *large. If you need to review the overall elements of Romik's approach and the context of these problems, you can look back at Problem 1 of Assignment 5.*

Consider for $R > 0$ *the curve*

$$C_R = \left\{ \left(\frac{1}{R^2} \right) p(Re^{it}) : 0 \le t \le 2\pi \right\},$$

parameterized by

$$\beta_R(t) = \frac{1}{R^2}\, p(Re^{it}) \qquad \text{for } 0 \le t \le 2\pi$$

and the parameterized curve $\beta_\infty : [0, 2\pi] \to \mathbb{C}$ *by*

$$\beta_\infty(t) = e^{2it}.$$

The curve C_R *is a kind of (normalized) "blow-down" of the image curve* Γ_∞*. The idea is that as* $R \nearrow \infty$*, the curve* $\Gamma_\infty = \Gamma_\infty(R)$ *is getting "very large" and "very close" to a double covered circle. The blow-down allows us to see what* Γ_∞ *looks like on a fixed visible scale in order to see the geometry of the limit.*

(a) *Show that*

$$\lim_{R \nearrow \infty} p(Re^{it}) = \infty$$

uniformly in $t \in [0, 2\pi]$.

(b) *Show*

$$\lim_{R \nearrow \infty} \|\beta_R - \beta_\infty\|_{C^k} = 0$$

uniformly in $t \in [0, 2\pi]$*. If you need to review the definition of* C^k *here and the associated norm, see Assignment 6 Problem 1.*

(c) *What does part* **(b)** *above tell you about the image* Γ_∞ *for* $R > 0$ *large?*

(d) *Prove that for* $R > 0$ *large enough,*

$$\int_\alpha \frac{1}{z} = 4\pi i \neq 0.$$

where $\alpha(t) = p(Re^{it})$.

Problem 2 *((S&S Chapter 2 Exercise 7) Let* $f : D_1(0) \to \mathbb{C}$ *be holomorphic and let* $\mathcal{W} = \{f(z) : z \in D_1(0)\}$ *be the image of* f*. Show*

$$\operatorname{diam} \mathcal{W} = \sup_{z,w \in D_1(0)} |f(z) - f(w)| \geq 2|f'(0)|. \tag{1}$$

Hint:

$$2f'(0) = \frac{1}{2\pi i} \int_{\zeta=\alpha} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}.$$

Can you show that if equality holds in (1) then $f(z) = a_1 z + a_0$ *for some* $a_0, a_1 \in \mathbb{C}$*?*

Problem 3 *(S&S Exercise 2.10) The Weierstrass approximation theorem says that any real valued continuous function on a closed interval in* $\mathbb{R}$ *can be uniformly approximated by a polynomial function (with real coefficients).*

(a) *Find a complex valued continuous function* $f : \overline{D_1(0)} \to \mathbb{C}$ *on the closed unit disk in* $\mathbb{C}$ *which cannot be uniformly approximated by a polynomial*

$$p(z) = \sum_{n=0}^{k} a_n z^n$$

with complex coefficients. Hint: Theorem 5.2 in S&S.

(b) *Find a real valued continuous function* $f : \overline{D_1(0)} \to \mathbb{C}$ *on the closed unit disk in* $\mathbb{C}$ *which cannot be uniformly approximated by a polynomial with complex coefficients.*

(c) *(extra) Show that any continuous real valued function* $u : \overline{B_1(\mathbf{0})} \to \mathbb{R}$ *defined on the unit disk* $\overline{B_1(\mathbf{0})} = \{(x, y) : x^2 + y^2 \leq 1\}$ *in* $\mathbb{R}^2$ *can be uniformly approximated by a polynomial*

$$q(x, y) = \sum_{i+j \leq k} a_{ij} x^i y^j$$

in two real variables x *and* y *and with real coefficients* a_{ij}*. Hint: Look up (and go through the proof of) the Stone-Weierstrass Theorem. Well, I said it was "extra."*

(d) *(extra—just for fun) If you're not familiar with multi-indices, look up* **multi-index notation** *and make sense of the following Taylor expansion for a function* $u : U \to \mathbb{R}$ *with* $U \subset \mathbb{R}^n$ *at the point* $\mathbf{x}_0 \in U$*:*

$$\sum_{|\beta|=0}^{\infty} \frac{D^\beta u(\mathbf{x}_0)}{\beta!} (\mathbf{x} - \mathbf{x}_0)^\beta.$$

Problem 4 *(S&S Exercise 2.11; Poisson Integral Formula) Let* $f : D_1(0) \to \mathbb{C}$ *be holomorphic.*

(a) *Show that for* $z \in D_1(0)\backslash\{0\}$ *and* $|z| < r < 1$

$$\int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - w} = 0$$

where $w = r^2/\bar{z}$.

(b) *Show that for* $z \in D_1(0)$,

$$f(z) = \frac{1}{2\pi i} \int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta} \ \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right)$$

where $\alpha(t) = re^{it}$ *for any* r *with* $|z| < r < 1$. *Hint(s): Use part* **(a)** *along with the Cauchy integral formula.*

Problem 5 *(S&S Exercise 2.12; Poisson Integral Formula) Let* $u \in C^2(B^1(\mathbf{0}))$ *be* **harmonic** *on* $B_1(\mathbf{0}) = \{(x,y) : x^2 + y^2 < 1\}$. *You should recall that this means* $u : B_1(\mathbf{0}) \to \mathbb{R}$ *is twice continuously differentiable and*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(a) *Show there exists a holomorphic function* $f : D_1(0) \to \mathbb{C}$ *with* $f = u + iv$. *Hint(s): If one had such a holomorphic function* f, *then one would have* $f' = 2\partial u/\partial z$. *Write down* $g = 2\partial u/\partial z$, *and show* g *has a primitive on* $D_1(0)$.

(b) *If* u *extends continuously to* $\overline{B_1(\mathbf{0})}$, *show*

$$u(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1 - |\mathbf{x}|^2}{1 - 2|\mathbf{x}|\cos(\arg z - t) + |\mathbf{x}|^2} \qquad \textit{for } |\mathbf{x}| < 1.$$

Notice that the integral is a convolution in the argument of $u(e^{it})$ *with*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

which is called the **Poisson kernel**.

Problem 6 *(Schwarz Reflection Principle) Let θ and ϕ be fixed in the interval $(0, \pi/2)$. Assume $f : \Omega \to \mathbb{C}$ is holomorphic in the sector*

$$\Omega = \{re^{it} : 0 < r < \infty,\ 0 < t < 2\theta\}$$

and satisfies

$$f(re^{i\theta}) \in \{\rho e^{i\phi} : \rho > 0\} \qquad \text{for} \qquad r > 0.$$

Finally, let

$$\Omega_1 = \{re^{it} : 0 < r < \infty,\ 0 < t < \theta\} \qquad \text{and} \qquad \Omega_2 = \{re^{it} : 0 < r < \infty,\ \theta < t < 2\theta\}.$$

Use the Schwarz reflection principle to find a formula for $f(z)$ when $z \in \Omega_1$ in terms of a value of $f(\zeta)$ for some $\zeta \in \Omega_2$. Hint: The answer I got is

$$f(z) = e^{2i\phi}\ \overline{f(\overline{z}e^{2i\theta})}.$$

Problem 7 *(S&S Exercise 2.15) Let $f : D_1(0) \to \mathbb{C}\backslash\{0\}$ be holomorphic and extend continuously to $\overline{D_1(0)}$ with*

$$|f(z)| = 1 \qquad \text{when} \qquad |z| = 1.$$

Show f is constant. Hint(s): Extend f to $\mathbb{C}\backslash\overline{D_1(0)}$ by

$$f(z) = \frac{1}{\overline{f(1/\overline{z})}}.$$

Show the extension is entire.