Assignment 5: Cauchy's Theorem(s)
Boundary Behavior of Power Series
Due Tuesday March 8, 2022

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Problem 1 *(Dan Romik's notes: The Fundamental Theorem of Algebra) The nominal objective of this problem is consideration of the fundamental theorem of algebra which states that any non-constant polynomial*

$$p(z) = \sum_{n=0}^{k} a_n z^n$$

with complex coefficients $a_0, a_1, \ldots, a_k$ has a complex root. A polynomial, of course, considered as a complex function is an entire (holomorphic) mapping. The more relevant utility of the problem for us is that it involves

(1) *some (interesting) considerations concerning general complex mappings, e.g.,* **continuity and multiplicity***,*

(2) *the careful consideration of the* **Riemann surfaces** *associated with the simple power polynomials $p(z) = z^k$,*

(3) *introducing* **deformations** *(or the homotopy) of curves,*

(4) *some interesting* **estimates** *involving complex numbers, and*

(5) *the* **winding number** *of a curve with respect to a point.*

It's a pretty impressive list. I anticipate that in order to do this in a reasonable way, we'll only be able to do it in small pieces over two or three assignments. This is the first piece.

Let us simplify all our considerations by noting that we can assume the polynomial p is **monic**, *i.e,, the leading coefficient a_k satisfies $a_k = 1$. Similarly, we can consider a preliminary simple case in which the constant term $a_0 = 0$. In this case, $z = 0$ is a root, so we can henceforth assume $a_0 \neq 0$.*

The general idea is pretty simple: We consider a simple explicit deformation of a circle in the domain:

$$h(t;r) = re^{it} \qquad \text{for} \qquad \textit{for } (t,r) \in [0,2\pi] \times [\epsilon, R]$$

where ϵ and R are positive numbers. We will find explicit values of ϵ and R later satisfying $\epsilon < R$. You can think of ϵ as "small" or close to zero and R as "large." Given this deformation, the composition $p \circ h$ also represents a continuous deformation of an "initial" curve Γ_0 parameterized by $\gamma_0(t) = p \circ h(t;\epsilon)$ to a final curve Γ_1 parameterized by $\gamma_1(t) = p \circ h(t;R)$. With this framework in hand, our strategy to prove the fundamental theorem of algebra is roughly as follows:

(i) Γ_0 *is a curve all the points of which are close to the point $a_0 \neq 0$, and this implies Γ_0 does not "wind around the origin."*

(ii) Γ_R *(for R large) is a curve which does "wind around the origin."*

(iii) *At some point in the second deformation $H = p \circ h$, there is some (t_0, r_0) for which $p(h(t_0;r_0)) = 0$, and thus, $h(t_0, r_0)$ is a root of p.*

A few notes about this outline: We'll want to get careful **estimates** *when we verify* **(i)** *and* **(ii)**. *In some cases, like in this first installment, we will want to see everything pretty explicitly or as explicitly as possible. Also, for parts* **(i)** *and* **(ii)** *we'll eventually need to understand precisely what it means for a curve to "wind around" the origin, and we'll need to know how to measure that. That's where the* **winding number** *comes in. Assertion* **(iii)** *may seem kind of obvious, but I would suggest that it's really not. Of course, I can't give you an example of a continuous deformation of $\gamma_0(t) = 2 + e^{it}$ to $\gamma_1(t) = e^{it}$ with no point mapping to the origin, but on the other hand, there is no kind of "intermediate value theorem" we can apply directly to this kind of question. (Maybe there should be, and you can become famous by formulating it.)*

Enough with the preliminaries; let's try to get down to work. We can imagine we are going to attempt a kind of tedious induction. The base case would involve

$$p(z) = a_0 + z$$

with $k = 1$.

(a) *Determine initial and terminal radii* ϵ *and* R *for which the argument (i.e., Romik's argument) works in the case* $k = 1$ *with* $p(z) = a_0 + z$.

(b) *Draw the image curves associated with the deformation* $a_0 + re^{it}$ *and find the values* (t_0, r_0) *corresponding to a solution. Hint: I guess you'll need a branch of the complex logarithm to do this.*

(c) *When* $k = 2$ *we have* $p(z) = a_0 + a_1 z + z^2$ *where* $a_0 \neq 0$. *Complete the square to show it is enough to find a root of a polynomial* $q(w) = b_0 + w^2$ *for appropriate choices of* b_0 *and* w.

(d) *Again in the case* $k = 2$, *if* $b_0 \neq 0$, *write*

$$q(w) = b_0[1 + (c_0 w)^2]$$

for an appropriate choice of $c_0 \neq 0$. *Plot The following curves*

- **(i)** $\{1 + (c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for* r *fixed with* $0 < r < 1/|c_0|$. *Hint:* $c_0 = |c_0|e^{i\,\mathrm{Arg}(c_0)}$.
- **(ii)** $\{1 + (c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for* r *fixed with* $r = 1/|c_0|$. *Find the root of* q *(corresponding to a particular value of* t*) in this case, and find the root of the original polynomial* p.
- **(iii)** $\{1 + (c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for* r *fixed with* $r > 1/|c_0|$.

(e) *Let* $\mathcal{R}$ *denote the Riemann surface for* z^2, *and consider the special polynomial* $q_0 : \mathbb{C} \to \mathcal{R}$ *by* $q_0(z) = z^2$ *as a function from* $\mathbb{C}$ *into its natural domain* $\mathcal{R}$. *Plot the image curve*

$$\left\{q_0\left(re^{it}\right) \in \mathcal{R} : 0 \leq t \leq 2\pi\right\}$$

for r *fixed with* $r > 0$.

(f) *Repeat part* **(d)** *with the following modification: Consider the curve as an image in a Riemann surface with two sheets and a branch cut at* $z = 1$. *Do you find two values of* t *for the curve of part* **(ii)**?

(g) *How can your plots of parts* **(d)** *and* **(f)** *be modified to plot the full images*

$$\{q(re^{it}) : 0 \leq t \leq 2\pi\}$$

for q?

(h) *Again in the case $k = 2$ as introduced in part* **(c)**, *if $b_0 = 0$, show that you can write*

$$p(z) = \frac{a_1^2}{4}(1 + c_0 z)^2$$

for an appropriate choice of c_0. Plot The following curves

- **(i)** $\{(1 + c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for r fixed with $0 < r < 1/|c_0|$.*
- **(ii)** $\{(1 + c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for r fixed with $r = 1/|c_0|$. Find the root of p (corresponding to a particular value of t) in this case.*
- **(iii)** $\{(1 + c_0 re^{it})^2 : 0 \leq t \leq 2\pi\}$ *for r fixed with $r > 1/|c_0|$.*

(i) *Repeat part* **(h)** *with the images in the Riemann surface $\mathcal{R}$ for z^2.*

(j) *Consider the polynomial $p(z) = a_0 + a_1 z + z^2$ under the assumptions*

- **(i)** $a_0 \neq 0$,
- **(ii)** $a_1 \neq 0$, *and*
- **(iii)** $a_1^2 \neq 4a_0$.

Show that under these assumptions you can write

$$p(z) = b_0 \left[\frac{a_1^2}{4b_0}(c_0 z + 1)^2 + 1 \right]$$

for appropriate choices of b_0 and c_0. Plot the image curve

$$\Gamma_r = \{p(re^{it}) : 0 \leq t \leq 2\pi\}$$

for r fixed.

(k) *Returning to the case $k = 1$ of parts* **(a)** *and* **(b)**, *calculate*

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z}$$

where $\gamma(t) = p(re^{it}) = a_0 + a_1 re^{it}$ for $0 \leq t \leq 2\pi$ with $r \neq r_0$.

Problem 2 *(Boundary Behavior and Mapping) Consider the complex tangent function defined by*

$$\tan z = \frac{\sin z}{\cos z}. \tag{1}$$

(a) *This is* **not** *an entire function. What is the domain Ω_p in $\mathbb{C}$ where* tan *is finite valued?*

(b) *What is a fundamental domain for* tan*, i.e., a domain $\Sigma \subset \mathbb{C}$ on which*

$$\tan : \Sigma \to \mathbb{C}$$

is one-to-one and onto?

(c) *Where are the branch points in the image of*

$$\tan : \Omega_p \to \mathbb{C}?$$

(d) *What is the image $\Omega \subset \Sigma$ of the unit disk $D_1(0)$ under the complex arctangent function?*

(e) *Denote by $\tan^{-1} : D_1(0) \to \Omega$, and prove that*

$$\frac{d}{dw}\tan^{-1} w = \frac{1}{1+w^2}.$$

Hint: $\tan(\tan^{-1} w) = w$ *and it's still true that*

$$\frac{d}{dz}\tan z = \sec^2 z$$

by the quotient rule.

(f) *Show* $\tan^{-1}$ *has a power series expansion (with radius of convergence $R = 1$) on $D_1(0)$. Hint: Find the series expansion for the derivative, then formally obtain a candidate series representing a holomorphic function $f : D_1(0) \to \mathbb{C}$ using termwise integration. You'll need something like Corollary 3.4 in Chapter 1 of Stein and Shakarchi to conclude $f(w) = \tan^{-1} w$ or, alternatively, Problem 10 of Assignment 4.*

Problem 3 *(Abel Limit Point Theorem; Problem 4 of Assignment 4) Go back and consider the series*

$$f(w) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, w^n \tag{2}$$

of Problem 4 of Assignment 4.

(a) *Find the image of* $D_1(0)$ *under* $f(w) = \log(1+w)$*.*

(b) *Use Corollary 3.4 in Chapter 1 of Stein and Shakarchi or, alternatively, Problem 10 of Assignment 4 to conclude* $f(w) = \log(1+w)$*. Hint:*

$$e^{\log(1+w)} = 1 + w.$$

(c) *It's always*[1] *worth repeating: What is the sum of the alternating harmonic series?*

Problem 4 *(Complex Logarithm)*

(a) *Explain how to define a function* $\arg : \mathbb{C} \to [0, 2\pi)$ *giving the argument of a complex number using a branch of the complex logarithm.*

(b) *Explain how to define a function* $\arg : \mathbb{C} \to (-\pi, \pi]$ *giving the argument of a complex number using a branch of the complex logarithm.*

(c) *The documentation for Mathematica says that the standard Mathematica implementation of the complex logarithm* `Log` *has a "branch cut discontinuity" in the complex plane running from* $-\infty$ *to* 0*. Can you guess what results from the Mathematica command*

```
Plot[Im[Log[Cos[t] + I Sin[t]]], {t, -2 Pi, 2 Pi}]
```

without actually executing the command? (Note: `Im` *is the standard function in Mathematica for taking the imaginary part of a complex number and* `I` *is the Mathematica notation for* $i \in \mathbb{C}$*.)*

(d) *How could you use the standard Mathematica commands* `Log` *and* `Im` *to create a function (in Mathematica) giving the argument of part (a) above?*

[1]The "three R's of learning" are "Repetition, repetition, and repetition."

(e) *Let* $\Sigma_j = \{x+iy \in \mathbb{C} : j\pi < y \leq (j+1)\pi\}$ *for* $j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$*, and let* $\log_j : \mathbb{C}\backslash\{0\} \to \Sigma_j$ *denote the branch of the complex logarithm corresponding to the strip* Σ_j *for* $j \in \mathbb{Z}$*. How could you use the standard Mathematica command* `Log` *to create a function (in Mathematica) giving the function* $\log_j$*? Incidentally,* `Log[-1]` *returns* $i\pi$ *in Mathematica.*

Problem 5 *(Complex Arctangent) The documentation for Mathematica says that the standard Mathematica implementation of the complex arctangent* `ArcTan` *has "branch cut discontinuities" in the complex plane running from* $-\infty i$ *to* $-i$ *and from* $+\infty i$ *to* i*.*

In addition the implementation `ArcTan` *can take alternatively* **two arguments** *so that*

`ArcTan[z,w]` *is equivalent to* `Im[Log[z+iw]]`.

What is the relation between `ArcTan[z,w]` *and* `ArcTan[z+iw]`*?*

Problem 6 *(S&S Exercise 2.1, Gaussian and Fresnel Integrals)*

(a) *Prove*

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Hint(s): This has nothing to do with complex analysis. Consider the square

$$\left(\int_0^R e^{-x^2}\, dx\right)^2 = \left(\int_0^R e^{-x^2}\, dx\right)\left(\int_0^R e^{-y^2}\, dy\right).$$

Write this as an iterated integral and then an area integral over an appropriate region in $\mathbb{R}^2$*. Change to polar coordinates. Evaluate and take the limit as* $R \nearrow \infty$*.*

(b) *Consider the real integrals*

$$\int_0^R \cos(x^2)\, dx \qquad \text{and} \qquad \int_0^R \sin(x^2)\, dx.$$

Explain why it might be reasonable to expect the limits

$$\lim_{R \nearrow \infty} \int_0^R \cos(x^2)\, dx \qquad \text{and} \qquad \lim_{R \nearrow \infty} \int_0^R \sin(x^2)\, dx$$

exist. Hint: Plot $\cos(x^2)$ *and* $\sin(x^2)$ *on* $[0, \infty)$*.*

(c) *Give explicit estimates in terms of a series to prove the limits*

$$\lim_{R\nearrow\infty}\int_0^R \cos(x^2)\,dx \qquad \text{and} \qquad \lim_{R\nearrow\infty}\int_0^R \sin(x^2)\,dx$$

exist. Hint(s): Break up $[0,\infty)$ *into intervals on which, for example,* $\cos(x^2)$ *maintains a single sign. Then estimate using an alternating series.*

(d) *Prove*

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

These are called **Fresnel integrals**. *Hint: Compute the complex integral*

$$\int_\alpha f$$

where $f(z) = e^{-z^2}$ *and* α *is a (counterclockwise) parameterization of the boundary of* $\Omega_R = \{re^{i\theta} : 0 < r < R,\ 0 < \theta < \pi/4\}$.

Problem 7 *(Goursat's Theorem) For this problem you should use Goursat's theorem as stated in S&S:*

Theorem 1 *(Goursat's theorem) If $f : \Omega \to \mathbb{C}$ is complex differentiable and $\mathcal{U}$ is a triangular domain with boundary a triangle*

$$T = \partial\mathcal{U} \qquad \text{with} \qquad T \cup \mathcal{U} = \overline{\mathcal{U}} \subset \Omega,$$

then

$$\int_\alpha f = 0$$

where $\alpha : [a, b] \to T$ is a parameterization of the triangle $T = \partial\mathcal{U}$.

(a) *Given three cyclically concatenated segments in the plane (real or complex) by which we mean there are three (distinct) points a, b and c with segments parameterized by*

$$\gamma(t) = (1-t)a + tb, \tag{3}$$
$$\alpha(t) = (1-t)b + tc, \ \textit{and} \tag{4}$$
$$\beta(t) = (1-t)c + ta, \tag{5}$$

each defined for $0 \le t \le 1$, these segments[2] always bound a triangular domain $\mathcal{U}$ with boundary a triangle. Give examples where four concatenated segments bound a (connected) quadrilateral domain, and give examples in which four concatenated segments bound something more complicated.

(b) *Use Goursat's theorem to prove the following:*

Theorem 2 *(Quadrilateral Goursat theorem) If $f : \Omega \to \mathbb{C}$ is complex differentiable and $\mathcal{V}$ is a connected quadrilateral domain with boundary a quadrilateral*

$$Q = \partial\mathcal{V} \qquad \text{with} \qquad Q \cup \mathcal{V} = \overline{\mathcal{V}} \subset \Omega,$$

then

$$\int_\alpha f = 0$$

where $\alpha : [a, b] \to Q$ is a parameterization of the quadrilateral $Q = \partial\mathcal{V}$.

[2]The parameterization α here is different from the parameterization α in the statement of Goursat's theorem above. Note that I'm loosely borrowing here notation from Euclidean/high school geometry in which the side of a triangle opposite a given vertex shares a letter: Side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C. For me the side parameterized by α is opposite the vertex a.

(c) *Formulate and prove a version of Theorem 2 for the case when there are four concatenated segments in Ω, but they do not bound a connected quadrilateral.*

Problem 8 *(Goursat subdomains) This problem is about the proof of Goursat's theorem; you should not use Goursat's theorem in your solution.*

Given an open set $\Omega \subset \mathbb{C}$, define a **simple Goursat subdomain** *to be an open set $\mathcal{U}$ for which the following hold:*

(i) $\overline{\mathcal{U}} \subset \Omega$,

(ii) $\partial\mathcal{U}$ *is a closed curve/contour.*[3]

(iii) *There exists a fixed natural number ν and there exists a fixed scale $\mu \in (0,1)$ such that*

$$\overline{\mathcal{U}} = \bigcup_{j=1}^{\nu} \overline{\mathcal{U}_j}$$

for some subdomains $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\nu \subset \mathcal{U}$ satisfying

(iv) $\mathcal{U}_k \cap \mathcal{U}_j = \phi$ *for $j \neq k$,*

(v)

$$\mathcal{H}^2\left(\overline{\mathcal{U}_j} \cap \overline{\mathcal{U}_k}\right) = 0 \qquad \text{for } j \neq k,$$

where $\mathcal{H}^2$ denotes area measure on the plane,

(vi) *Each $\mathcal{U}_j$ is geometrically similar to $\mathcal{U}$ with*

$$\mathcal{U}_j = \{\mu z + w_j : z \in \mathcal{U}\}$$

for some $w_j \in \mathbb{C}$, $j = 1, 2, \ldots, \nu$, and

(vii)

$$\int_\alpha f = \sum_{j=1}^{\nu} \int_{\alpha_j} f \tag{6}$$

where α is a counterclockwise parameterization of $\partial\mathcal{U}$ and α_j is a counterclockwise parameterization of $\partial\mathcal{U}_j$.

[3] By "contour" we mean a curve admitting a piecewise regular parameterization—a curve Γ that can be used to construct a complex integral of a continuous function $f : \Gamma \to \mathbb{C}$.

The trianglular domain is an example of a Goursat subdomain with $\nu = 4$ and $\mu = 1/2$. The rectangular domain is also an example of a Goursat subdomain with $\nu = 4$ and $\mu = 1/2$. Ahlfors proves Goursat's theorem for a rectangular subdomain.

(a) *Give an example of a Goursat subdomain with $\nu \neq 4$ and/or $\mu \neq 1/2$.*

(b) *Give an example of a Goursat subdomain which is not a triangular domain or a rectangular domain.*

(c) *Can you prove either of properties* **(v)** *and/or* **(vii)** *of a simple Goursat subdomain follow from the other properties in the definition?*

(d) *If $\mathcal{U}$ is a simple Goursat subdomain in an open set $\Omega \subset \mathbb{C}$, then each set $\mathcal{U}_j$ for $j = 1, 2, \ldots, \nu$ is a Goursat subdomain in Ω.*

(e) *Prove Goursat's theorem in the following form:*

> ***Theorem 3*** *(Goursat's theorem) If $f : \Omega \to \mathbb{C}$ is complex differentiable and $\mathcal{U}$ is a simple Goursat subdomain with respect to Ω, then*
>
> $$\int_\alpha f = 0$$
>
> *where $\alpha : [a, b] \to \Omega$ is a parameterization of $\partial\mathcal{U}$ and $f : \Omega \to \mathbb{C}$ is holomorphic.*

Problem 9 *(Goursat subdomains) Given an open set $\Omega \subset \mathbb{C}$, define a* **general Goursat subdomain** *to be a domain $\mathcal{U} \subset \mathbb{C}$ for which $\partial\mathcal{U}$ is a closed contour and $\overline{\mathcal{U}} \subset \Omega$ with*

$$f : \Omega \to \mathbb{C} \text{ holomorphic} \qquad \Longrightarrow \qquad \int_\alpha f = 0$$

where α parameterizes $\Gamma = \partial\mathcal{U}$ and $f : \Omega \to \mathbb{C}$ is any holomorphic function.

(a) *Prove the following result:*

> ***Theorem 4*** *The conformal image of a general Goursat subdomain is a general Goursat subdomain in the following sense: If $\mathcal{U}$ is a general Goursat subdomain in Ω and*
>
> **(i)** *$\phi : \Omega \to W$ is a surjective holomorphic function onto an open set $W \subset \mathbb{C}$,*

(ii) $\phi(\overline{\mathcal{U}}) = \overline{\mathcal{V}}$ *for some open set* $\mathcal{V} \subset W$ *with* $\partial\mathcal{V}$ *a closed contour with parameterization* $\beta = \phi \circ \alpha$ *with* α *a parameterization of* $\partial\mathcal{U}$ *as in the definition,*

then

$$\int_\beta f = \int_\alpha f \circ \phi \ \phi' = 0$$

for any holomorphic function $f : W \to \mathbb{C}$*.*

(b) *Prove any simple Goursat subdomain is a general Goursat subdomain.*

(c) *Prove a rectanglular domain* $\mathcal{R} = \{x + iy : 0 < x < a,\ 0 < y < b\}$ *in standard position with respect to any open set containing*

$$\mathcal{R}_\epsilon = \{x + iy : -\epsilon < x < a + \epsilon,\ -\epsilon < y < b + \epsilon\}$$

for any $\epsilon > 0$ *is both a simple and a general Goursat subdomain.*

(d) *Given any open set* $\Omega \subset \mathbb{C}$*, prove any quadrilateral* $\mathcal{Q}$ *with* $\overline{\mathcal{Q}} \subset \Omega$ *is a general Goursat subdomain with respect to* Ω*.*

Problem 10 *(Goursat's theorem with extra regularity; S&S Chapter 2 Exercise 5) This is about the proof of Goursat's theorem, so you should not use Goursat's theorem to solve the problem.*

In my notes on integration I discuss first the integral of a (continuous) real valued function $f : \Gamma \to \mathbb{R}$ defined on a regular curve $\Gamma \subset \mathbb{R}^2$. Such an integral on a curve requires no orientation, but both the integrand in

$$\int_\Gamma f$$

may have a special form involving orientation and the curve may involve orientation. Two important examples of this involve a simple closed curve Γ *which, by the Jordan curve theorem is the boundary of a unique bounded open set $U \subset \mathbb{R}^2$ and a vector field which is a function $\mathbf{v} : \overline{U} \to \mathbb{R}^2$. The vector field will also be assumed to be continuous, and the component functions will be assumed to be continuously differentiable. The first example of an integrand of special form is called a* **flux integral***:*

$$\int_{\partial U} \mathbf{v} \cdot \mathbf{n}$$

where $\mathbf{n}$ is the outward unit normal to ∂U. Note that one still integrates on the curve here as a set, but there is an implied orientation by the choice of the normal. Associated with flux integrals is Gauss' theorem (or the **divergence theorem***) in the plane which states that*

$$\int_{\partial U} \mathbf{v} \cdot \mathbf{n} = \int_U \operatorname{div} \mathbf{v}.$$

The divergence operator can be given a nice invariant definition, but for our purposes we can simply say

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

where $\mathbf{v} = (v_1, v_2)$.

The second "special form" integral is a **circulation integral***:*

$$\int_{\partial U} T \cdot \mathbf{v}$$

where T is a counterclockwise unit tangent vector—the tangent vector of a counterclockwise arclength parameterization—on ∂U. Associated with these integrals is Green's theorem which states

$$\int_{\partial U} T \cdot \mathbf{v} = \int_U \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right).$$

(a) *Show the divergence theorem in the plane implies Green's theorem.*

(b) *Show Green's theorem implies the divergence theorem in the plane.*

(c) *Use the divergence theorem in the plane to show the following:*

> ***Theorem 5*** *(Goursat's theorem for the bounded component of the complement of a simple closed curve) Given $f : \Omega \to \mathbb{C}$ holomorphic and $\overline{\mathcal{U}} \subset \Omega$ where $\mathcal{U}$ is the bounded component of the complement of a simple closed curve $\Gamma = \partial\Omega$ with $f = u + iv$ as usual with $u, v \in C^1(\overline{\mathcal{U}})$ we have*
>
> $$\int_\alpha f = 0$$
>
> *where α is a (counterclockwise) parameterization of Γ.*

(d) *Use Green's theorem to prove Theorem 5 above.*

Note(s) on regularity: The quadrasection proof of Goursat's theorem does not require continuity of the partial derivatives of the real and imaginary parts of f. Technically, when we say $u, v \in C^1(\overline{\mathcal{U}})$ what we mean is that there is a larger open set containing $\overline{\mathcal{U}}$ and u and v are C^1 functions on this larger open set. Certainly $u, v \in C^1(\Omega)$ would imply this condition.