Assignment 4: Chapter 1 (exam)
Complex Integration (and other topics)
Due Tuesday February 22, 2022

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Problem 1 *(S&S 1.16(f) Bessel Functions) Bessel's ordinary differential equation (if I can type it correctly) is*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{1}$$

In general, this is called Bessel's ODE of order ν*, and the natural context in which to consider the equation is for a complex valued function* $y : (0, \infty) \to \mathbb{C}$ *and with* ν *a fixed complex number. Though you may not have considered complex valued solutions* $y : (a, b) \to \mathbb{C}$ *to ordinary differential equations before, they are quite natural to consider. Such a consideration, furthermore, doesn't immediately have anything to do with complex analysis; you're just using complex numbers. If you attempt to extend a complex valued function of a real variable to an open subset* Ω *of* $\mathbb{C}$*, then you are doing complex analysis, and that's what we are going (to try) to do here.*

Notice that the Bessel equation (1) is singular at $x = 0 \in \mathbb{R}$*. A Froebenius series solution is a function of the form*

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}. \tag{2}$$

Notice that if $\alpha = 0$*, this is just a power series solution. More generally if* $\alpha \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$*, then the Froebenius solution is just a power series solution with perhaps some number of zero coefficients at the beginning. This is the special case we are going to use.*

(a) *Assume $\nu \in \mathbb{N}_0$ and attempt to find a Froebenius series solution of the Bessel equation. Hint(s): Plug in a series of the form (2) and differentiate the series termwise freely. Assume the coefficients of various powers of x must vanish and find a polynomial equation for α. This equation is called the* **indicial equation** *in the method of Froebenius. That is,*

(i) *Plug the series (2) into the ODE (1) and set the coefficient of the lowest order term appearing on the left equal to zero. This should be the indicial equation. Choose a solution α of the indicial equation in $\mathbb{N}_0$.*

(ii) *With your choice of α from* **(i)** *proceed to set the coefficients of higher powers of x equal to zero and obtain* **recursion relations** *between the coefficients $a_0, a_1, a_2, a_3, \ldots$.*

(iii) *For each $\nu \in \mathbb{N}_0$ you should get at least one power series of the form (2) with some $\alpha \in \mathbb{N}_0$ that formally solves the Bessel ODE. Classify all solutions you find.*

(b) *Consider the (formal) series*

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\nu)!} \left(\frac{z}{2}\right)^{2m+\nu}. \tag{3}$$

How does this series fit into the formal solutions you found in part (a)?

(c) *The function defined by the complex series (3) is called the* **Bessel function** *(of integer order ν)* **of the first kind***. Find the radius of convergence of $J_\nu(z)$.*

Problem 2 *(S&S 1.19(a)) Consider the formal power series*

$$\sum_{n=0}^{\infty} n\, z^n. \tag{4}$$

(a) *Find the radius of convergence R of the series.*

(b) *Analyze the convergence of the series (4) for $|z| = R$.*

(c) *Identify the differentiable function $f : B_R(0) \to \mathbb{C}$ represented by the series (4). Hint(s):*

(i) *We haven't shown anything about complex Taylor expansions, but assume*

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\, z^n$$

is valid for a complex differentiable function $g : B_r(0) \to \mathbb{C}$ defined on a disk.

(ii) *What coefficients do you get for the geometric series*

$$g(z) = \frac{1}{1-z}?$$

(iii) *What coefficients do you need for f?*

Problem 3 *(S&S 1.23, real analyticity) We are (eventually) going to show that a (complex) differentiable function is repeatedly differentiable (as many times as we would like to differentiate it). That is, for complex differentiability*

$$\textit{once differentiable} \qquad \Longrightarrow \qquad \textit{infinitely differentiable.}$$

We will also show that a (complex) differentiable function is always locally represented by a power series (and that power series is the Taylor series at a given point $z_0 \in \Omega$).

The point of this exercise is that these assertions are not valid with respect to real differentiation—though most people don't run across (or think much about) counterexamples every day.

(a) *Find a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^k(\mathbb{R})\backslash C^{k+1}(\mathbb{R})$ for every $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ where $C^k(a, b)$ is the collection of all real valued functions having k continuous derivatives well defined on the open interval $(a, b) \subset \mathbb{R}$.*

(b) *Consider $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0. \end{cases}$$

(i) *Show $f \in C^\infty(\mathbb{R})$, that is every derivative $f^{(n)}(x)$ exists for every $n \in \mathbb{N}_0$ and every $x \in \mathbb{R}$.*

(ii) *Show that $f \notin C^\omega(\mathbb{R})$, that is, there exists some $x_0 \in \mathbb{R}$ such that there is* **no power series**

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

representing f, i.e., convergent to f with

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

in any neighborhood $(x_0 - \epsilon, x_0 + \epsilon)$.

(c) *Let a and b be real numbers with $0 < a < b$. Find a function $u : \mathbb{R}^n \to \mathbb{R}$ with $u \in C^\infty(\mathbb{R}^n)$, which means that each and every partial derivative of u you might ever imagine to calculate exists and is continuous on all of $\mathbb{R}^n$* **and**

$$u\Big|_{B_a(\mathbf{0})} \equiv 1 \qquad \text{and} \qquad u\Big|_{\mathbb{R}^n \backslash B_b(\mathbf{0})} \equiv 0.$$

Problem 4 *(S&S Exercise 1.14, 15, 19(c))*

(a) *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of complex numbers. Set*

$$S_k = \sum_{j=1}^{k} b_j.$$

That is, $S_k \in \mathbb{C}$ *is the* k*-th partial sum of the formal series* $\sum b_j$ *associated with the second sequence. Show that for any* $M \in \{2, 3, 4, \ldots\}$ *and any* $N \in \{3, 4, 5, \ldots\}$ *with* $N > M$ *there holds*

$$\sum_{n=M}^{N} a_n\, b_n = a_N\, S_N - a_M\, S_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n)\, S_n.$$

This is what Stein calls the **summation by parts** *formula. Hint: It's easy to prove by induction on* N*.*

(b) *Prove a second summation by parts formula for sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$*. This time set*

$$S_k = \sum_{j=0}^{k} b_j.$$

and show that for any $M \in \mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *and any* $N \in \{2, 3, 4, 5, \ldots\}$ *with* $N > M$ *there holds*

$$\sum_{n=M}^{N} a_n\, b_n = a_N\, S_N - a_M\, S_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n)\, S_n.$$

(c) **Abel Limit Theorem** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers and assume the associated series*

$$\sum_{j=0}^{\infty} a_n \qquad \textbf{converges} \textit{ to a complex number } w \in \mathbb{C}.$$

Show the following:

(i) *For each* $x \in (0, 1)$ *the series*

$$\sum_{n=0}^{\infty} a_n x^n \qquad \textit{converges as well.}$$

(ii)

$$\lim_{\mathbb{R}\ni x\nearrow 1} \sum_{n=0}^{\infty} a_n x^n = w = \sum_{j=1}^{\infty} a_n.$$

Hint(s): For part **(ii)** *first use the summation by parts formula of part* **(b)** *with $M = 1$ to write the convergent series in part* **(i)** *in a different form:*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.$$

Note there are two ways to apply the summation by parts formula. (You have to figure out which way to apply it, and you might try both ways to see what you get.)
Finally, show

$$\sum_{n=0}^{\infty} b_n x^n - w = (1-x) \sum_{n=0}^{\infty} (c_n - w) x^n.$$

Break this quantity into two pieces

$$(1-x) \sum_{n=0}^{N} (c_n - w) x^n + (1-x) \sum_{n=N+1}^{\infty} (c_n - w) x^n.$$

Prove the whole thing goes to zero as follows: Let $\epsilon > 0$. Use the fact that $c_n \to w$ to make the second part smaller than $\epsilon/2$ for a fixed N. Then with N fixed show that if x is close enough to 1, the first part is smaller than $\epsilon/2$ as well.

(d) *Consider the formal power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n. \tag{5}$$

(i) *Find the radius of convergence R of the series.*

(ii) *Analyze the convergence of the series (5) for $|z| = R$.*

(iii) *Identify the differentiable function $f : B_R(0) \to \mathbb{C}$ represented by the series (5). Hint: Look back at Problem 2.*

(e) *Find the value of the alternating harmonic series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Problem 5 *(real differentiability) If* $\alpha : [a,b] \to \mathbb{C}$ *with* $x = \operatorname{Re}\alpha$*,* $y = \operatorname{Im}\alpha$*, and* $x, y \in C^1[a,b]$*, then (show that) for each* $t \in (a,b)$ *there is a complex number* $L \in \mathbb{C}$ *for which*

$$\lim_{h \to 0} \frac{\alpha(t+h) - \alpha(t)}{h} = L.$$

Note that $h \in \mathbb{R}$ *in this limit. What is the limit* L*?*

Problem 6 *(change of variables) Verify the following concerning change of variables: If*

(i) $\alpha : [a,b] \to \Gamma$ *parameterizes a curve* $\Gamma \subset \mathbb{C}$*,*

(ii) $\beta : [c,d] \to \Gamma$ *parameterizes the same curve,*

(iii) $\xi : [a,b] \to [c,d]$ *is a change of variables, and*

(iv) $g : \Gamma \to \mathbb{C}$ *is continuous,*

then

(a) *The usual chain rule holds for the composition of a complex valued function of a real variable and a real valued function of a real variable:*

$$(\beta \circ \xi)' = (\beta' \circ \xi)\ \xi'.$$

(b) *The usual change of varibles formula holds for the (hybrid) integral of a complex valued function on a complex curve subject to a real change of variable:*

$$\int_a^b g(\beta \circ \xi(t))\ \xi'(t)\, dt = \int_c^d g \circ \beta(\xi)\, d\xi.$$

Problem 7 *(Exercises related to the proof of Theorem 3.2 in S&S)*

(a) *If* $g : [a, b] \to \mathbb{C}$ *has continuous (real) derivative* $g' = h' + ik'$*, then*

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

This is a version of the fundamental theorem of calculus for complex valued functions $g \in C^1([a, b] \to \mathbb{C})$*.*

(b) *(another chain rule) If* $f : \Omega \to \mathbb{C}$ *is differentiable and* $\alpha : [a, b] \to \Omega$ *has continuous (real) derivative* $\alpha' = x' + iy'$*, then*

$$\frac{d}{dt}(f \circ \alpha) = f' \circ \alpha \ \frac{d}{dt}\alpha.$$

Problem 8 *(a complex integral) Let* Γ *be the boundary of the square* $U = \{z = x + iy : 1 < x, y < 2\}$*. Compute*

$$\int_\gamma \frac{1}{z}.$$

Problem 9 *(S&S Exercise 1.25(a)) Let* $\alpha(t) = re^{it}$ *for* $r > 0$ *and* $0 \leq t \leq 2\pi$ *orient the boundary of the disk* $D_r(0) \subset \mathbb{C}$*.*

(a) *Find an arclength parameterization* γ *of* $\partial D_r(0)$*.*

(b) *Compute*

$$\int_\alpha z^n$$

where $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$*.*

(c) *Given complex numbers* $a, b \in \mathbb{C}$ *with* $|a| < r < |b|$*, compute*

$$\int_\alpha \frac{1}{(z-a)(z-b)}.$$

Problem 10 *(S&S Exercise 1.26) Let* $f : \Omega \to \mathbb{C}$ *and* $g : \Omega \to \mathbb{C}$ *be complex differentiable functions with* $f' = g'$ *on* Ω*. What can you say about the relation between* f *and* g*?*