Assignment 3:
Power Series
Due Tuesday February 8, 2022

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Problem 1 *(Antiholomorphic Functions) Remember that given a function* $f : \Omega \to \mathbb{C}$ *with* $f = u + iv$ *and* $u, v \in C^1(U)$*, we say* f *is holomorphic on* Ω *if*

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] = 0.$$

This condition is equivalent to the existence of the limit(s)

$$\lim_{\mathbb{C}\ni h\to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{C}$$

for every $z \in \Omega$.

Given $g : \Omega \to \mathbb{C}$ *with* $g = \tilde{u} + i\tilde{v}$ *and* $\tilde{u}, \tilde{v} \in C^1(U)$*, we say* g *is* **antiholomorphic** *on* Ω *if*

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}\right)\right] = 0 \qquad \text{on } \Omega.$$

(a) *Give a condition for* g *to be antiholomorphic in terms of the existence of the limit of a "difference quotient."*

(b) *Given an antiholomorphic function* $g = \tilde{u} + i\tilde{v}$ *as above, compute the value of the limit of the "difference quotient" you defined in the previous part.*

Problem 2 *(Daniel's Antiholomorphic Functions) Recall that the function* $h : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ *by* $h(z) = 1/z$ *is holomorphic on the punctured plane and has values that can be written in the form*

$$h(z) = \frac{\bar{z}}{|z|^2}.$$

Let $g : \mathbb{C} \to \mathbb{C}$ *be given by* $g(z) = \bar{z}$.

(a) *Find another antiholomorphic function* f *for which* $h = f \circ g = g \circ f$.

(b) *Choose a point* $z_0 \in \{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) > 0\}$ *and consider the curves* $\gamma_1(t) = z_0 + t$ *and* $\gamma_2(t) = z_0 + it$ *for* $0 \le t \le 1$ *along with the square* $C = \{z_0 + s + it : 0 \le s \le 1\}$. *Draw the following:*

- **(i)** *The images of* γ_1 *and* γ_2 *and the square* C.
- **(ii)** *The images of* $f \circ \gamma_1$, $f \circ \gamma_2$, *and* $f(C)$.
- **(iii)** *The images of* $g \circ \gamma_1$, $g \circ \gamma_2$, *and* $g(C)$.
- **(iv)** *The images of* $h \circ \gamma_1$, $h \circ \gamma_2$, *and* $h(C)$.

You may wish to use mathematical software (Something like Matlab, Mathematica, Maple, Octave, ...) in order to get a very accurate picture.

Problem 3 *(Linear Mappings of* $\mathbb{R}^2$*) Let* $L : U = \mathbb{R}^2 \to \mathbb{R}^2$ *by* $L(x, y) = (u(x, y), v(x, y))$ *be linear. Consider* $f : \Omega = \mathbb{C} \to \mathbb{C}$ *with* $f = u + iv$.

(a) *Classify all linear maps* L *for which* f *is holomorphic.*

(b) *Classify all linear maps* L *for which* f *is antiholomorphic.*

Problem 4 *(S&S Exercise 1.12) Consider* $f : \mathbb{C} \to \mathbb{C}$ *by* $f(x + iy) = \sqrt{|xy|}$. *Show that* f *satisfies the Cauchy-Riemann equations at* $z = x + iy = 0$, *but* f *is not (complex) differentiable at* $z = 0$.

Problem 5 *(S&S Exercise 1.13) Show that the following are equivalent for a holomorphic function* $f : \Omega \to \mathbb{C}$ *with* $f = u + iv$ *and* $u, v \in C^1(U)$*:*

(i) f *is constant on the open set* Ω.

(ii) $f(\Omega) = \{f(z) : z \in \Omega\}$ *is a subset of a line.*

(iii) $f(\Omega) = \{f(z) : z \in \Omega\}$ *is a subset of a circle.*

Hint: Consider the case where $f(\Omega)$ lies in a vertical line.

Problem 6 *(S&S Exercise 1.16) Find the radius of convergence of the following series:*

(a)

$$\sum_{n=0}^{\infty} (\log n)^2 \ z^n.$$

(b)

$$\sum_{n=0}^{\infty} n! \ z^n.$$

(c)

$$\sum_{n=0}^{\infty} \frac{n^2}{4^n + 3n} \ z^n.$$

Problem 7 *(S&S Exercise 1.17) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of coefficients in $\mathbb{C}$. If*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists as a real number in $(0, \infty)$, then (show) this is the Hadamard radius. Note: the quantity R here is the reciprocal of the quantity one would get from the ratio test. Hint: Stackexchange.

Problem 8 *(S&S Exercise 1.16) Find the radius of convergence of the hypergeometric series:*

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!} \ z^n$$

where

$$c_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}.$$

Hint: Use the ratio test.

Problem 9 *(S&S Exercise 1.18) Consider a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence $R > 0$. Show that f has a (convergent) power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

with center z_0 for any $z_0 \in D_R(0)$.

Problem 10 *(S&S Exercise 1.20, Ahlfors Exercise 2.4.1) For each positive integer $m = 1, 2, 3, \ldots$, find the power series expansion for the function*

$$f(z) = \frac{1}{(1 - z)^m}$$

with center at the origin, and find the radius of convergence.