Assignment 2 Problem 7
Partial Solution

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Stein suggests using the alternative definition[1] of complex differentiability to prove the differentiability and (familiar) derivative formula/chain rule

$$(g \circ f)'(z) = g' \circ f(z)\ f'(z)$$

for the composition of two holomorphic functions f and g, and I have assigned this as part of Assignment 2 Problem 7. Stein says "one proves easily" this formula, but I am guessing if this kind of argument is not familiar from advanced calculus, then it may not be easy. In any case, I'm going to try to prove it (carefully as usual), and maybe even correctly.

It may be noted that the following argument has essentially nothing to do with complex analysis, and it would be precisely the same argument if one wanted to show the same formula holds for the composition $g \circ f$ of real functions defined on an open interval. Only the metric changes.

Here is how I would prove it: First I would write down what I want to prove. That is,

$$(g \circ f)(z+h) - (g \circ f)(z) - g' \circ f(z)\ f'(z)\ h = \circ(h). \tag{1}$$

This means precisely that

$$\lim_{h \to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z) - g' \circ f(z)\ f'(z)\ h}{h} = 0,$$

or for any $\epsilon > 0$, there is some $\delta > 0$ for which $0 < |h| < \delta$ implies

$$\left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - g' \circ f(z)\ f'(z)\ h}{h} \right| < \epsilon. \tag{2}$$

To this end, let $\epsilon > 0$. Since g is differentiable at $f(z)$, we know

$$g(f(z)+k) - g(f(z)) - g'(f(z))k = \circ(k).$$

In particular, there is some $\delta_1 > 0$ for which $0 < |k| < \delta_1$ implies

$$\left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))k}{|k|} \right| < \frac{\epsilon}{2}\ \frac{1}{2|f'(z)|+1}. \tag{3}$$

I must confess at this point, that I had to look ahead at the proof to decide on the positive number

$$\frac{\epsilon}{2}\ \frac{1}{2|f'(z)|+1}$$

[1] Alternative to the difference quotient definition that is.

appearing on the right in (3). From the point of view of reading the proof (or writing a proof) however, all I need to know is that this is some positive number, and hopefully that's clear. It turns out that I don't have to go very far into the main estimation to understand why I chose that number. Note the relation of the numerator on the left in (3) to the first two terms in the numerator on the left in (2): I can write

$$(g \circ f)(z+h) - (g \circ f)(z) = (g(f(z) + [f(z+h) - f(z)]) - g(f(z))$$

which is the same as $g(f(z)+k) - g(f(z))$ for the particular value $k = f(z+h) - f(z)$. More generally, when I estimate the left side in (2) I'm going to use that value of k and the triangle inequality to write

$$\begin{aligned}
\left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - g' \circ f(z)\ f'(z)\ h}{h} \right| &= \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{h} \right| \\
&\quad + \left| \frac{g'(f(z))\ k - g'(f(z))\ f'(z)\ h}{h} \right| \\
&\leq \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{h} \right| \\
&\quad + |g'(f(z))| \left| \frac{k - f'(z)\ h}{h} \right| \\
&\leq \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{h} \right| \\
&\quad + |g'(f(z))| \left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right|.
\end{aligned}$$

Thus, to get my final estimate (2) I'm going to estimate two terms and show each of them is smaller than $\epsilon/2$. Now, look at the first term

$$\left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{h} \right|. \tag{4}$$

I want to use (3) to get an estimate on this term, and I have in the back of my mind writing this term as

$$\left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{k} \ \frac{k}{h} \right|. \tag{5}$$

Of course, I can't actually write this because it may very well be the case that $k = f(z+h) - f(z)$ is zero. But note that when $k = 0$, the quantity in (4) also vanishes, so it is surely less than $\epsilon/2$. In the case when $k \neq 0$, I can write (5), and I can also write it as

$$\left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{k} \right| \ \left| \frac{f(z+h) - f(z)}{h} \right|.$$

Hopefully, the first factor here is going to be handled using (3) or some estimate like (3). But when I get done, I'm going to have a second factor to deal with, namely,

$$\left| \frac{f(z+h) - f(z)}{h} \right|.$$

Now I note that I'm going to be able to make h small, and if I do, this term should get close to $|f'(z)|$. I may very well not be able to make it smaller than $|f'(z)|$, but a first good thought might be that I might be able to make it smaller than twice $|f'(z)|$. Now that will probably work, more or less, except that I may have trouble if $2|f'(z)| = |f'(z)| = 0$, which could very well happen. Hence the quantity $2|f'(z)|+1 \geq 1 > 0$

appears. So I want something appearing on the right of (3) which is small enough so that when I multiply it by $2|f'(z)|+1$ I get something smaller than $\epsilon/2$. This explains the choice leading to (3).

Of course, all this scheming assumes I'll be able to choose h close enough to zero so that, first of all $k = f(z+h) - f(z)$ is close enough to zero so that (3) holds and also so that $|k/h| < 2|f'(z)| + 1$. So we have to worry about those things. Let's do that now.

The differentiability of f at z tells us that there is some $\delta_2 > 0$ for which $0 < |h| < \delta_2$ implies

$$\left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right| < \min\left\{ m_2,\ \frac{\epsilon}{2}\ \frac{1}{|g'(f(z))|+1} \right\}. \tag{6}$$

In this expression m_2 is any fixed positive number. I'm going to identify m_2 later, so we can just think of (6) as a preliminary version of what we are eventually going to conclude from the differentiability of f at z. The appearance of the quantity

$$\frac{\epsilon}{2}\ \frac{1}{|g'(f(z))|+1}$$

on the right of (6) should be pretty clear from the second term I'm planning to use to get the estimate in (2) namely

$$|g'(f(z))| \left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right|.$$

Let us turn then to the question of what is implied about $k = f(z+h) - f(z)$ by the inequality

$$\left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right| < m_2. \tag{7}$$

I have in mind now that differentiability implies continuity, and I'm expecting to somehow reconstruct some implication of that fact here. In particular, I need to assert that

$$|k| = |f(z+h) - f(z)| < \delta_1 \tag{8}$$

in order to use the conclusion of (3). In addition, I need

$$\frac{|k|}{|h|} = \left| \frac{f(z+h) - f(z)}{h} \right| < 2|f'(z)| + 1. \tag{9}$$

Starting from (7) I find

$$|k| = |f(z+h) - f(z)| \le |f(z+h) - f(z) - f'(z)\ h| + |f'(z)|\ |h| \le m_2|h| + |f'(z)|\ |h|.$$

In particular, if[2]

$$m_2 < |f'(z)| + 1 \qquad \text{and} \qquad |h| \le \frac{\delta_1}{2|f'(z)|+1}, \tag{10}$$

then

$$|k| < (m_2 + |f'(z)|)|h| < \delta_1$$

as desired. Notice that the same values also give

$$\frac{|k|}{|h|} = \left| \frac{f(z+h) - f(z)}{h} \right| < 2|f'(z)| + 1.$$

Thus, it appears we are ready to make the final argument/estimate. Let $m_2 > 0$ be chosen with $m_2 < |f'(z)| + 1$. I can then take δ_2 according to which (6) holds. Thus, for the ultimate specification

$$0 < |h| < \delta = \min\left\{ \frac{\delta_1}{2|f'(z)|+1}, \delta_2 \right\}, \tag{11}$$

[2]Notice that m_2 does not depend on δ_2 here.

we have that (7) holds and hence (8) and (9). Consequently, (3) holds, and we can estimate the first term

$$\left|\frac{g(f(z)+k)-g(f(z))-g'(f(z))\ k}{h}\right|$$

in our initial main equality

$$\left|\frac{(g\circ f)(z+h)-(g\circ f)(z)-g'\circ f(z)\ f'(z)\ h}{h}\right| \le \left|\frac{g(f(z)+k)-g(f(z))-g'(f(z))\ k}{h}\right| + |g'(f(z))|\left|\frac{f(z+h)-f(z)-f'(z)\ h}{h}\right|$$

in two cases. When $k=0$,

$$\left|\frac{g(f(z)+k)-g(f(z))-g'(f(z))\ k}{h}\right| = 0 < \frac{\epsilon}{2}.$$

If $k \neq 0$, I can write as in (5)

$$\begin{aligned}\left|\frac{g(f(z)+k)-g(f(z))-g'(f(z))\ k}{h}\right| &= \left|\frac{g(f(z)+k)-g(f(z))-g'(f(z))\ k}{k}\ \ \frac{k}{h}\right| \\ &< \frac{\epsilon}{2}\ \frac{1}{2|f'(z)|+1}\ (2|f'(z)|+1) \\ &= \frac{\epsilon}{2}.\end{aligned}$$

We have used here (8) and then (3) and finally (9).

The second term

$$|g'(f(z))|\left|\frac{f(z+h)-f(z)-f'(z)\ h}{h}\right|$$

on the right in the main inequality, is estimated directly using (6) with

$$|g'(f(z))|\left|\frac{f(z+h)-f(z)-f'(z)\ h}{h}\right| \le |g'(f(z))|\ \frac{\epsilon}{2}\ \frac{1}{|g'(f(z))|+1} < \frac{\epsilon}{2}.$$

The main inequality then yields that for

$$0 < |h| < \delta = \min\left\{\frac{\delta_1}{2|f'(z)|+1}, \delta_2\right\}$$

$$\left|\frac{(g\circ f)(z+h)-(g\circ f)(z)-g'\circ f(z)\ f'(z)\ h}{h}\right| < \epsilon.$$

We have shown

$$(g\circ f)(z+h)-(g\circ f)(z)-g'\circ f(z)\ f'(z)\ h = \circ(h). \qquad \square$$