Assignment 2:
Topology and Complex Differentiability
Due Tuesday February 1, 2022

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Problem 1 *(open sets Stein pages 5-6) Remember that an* **open disk** *with center* $z_0 \in \mathbb{C}$ *and radius* $r > 0$ *is a set of the form*

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\},$$

and a set $U \subset \mathbb{C}$ *is* **open** *if for every point* $z \in U$ *there is some* $r > 0$ *for which the open disk* $D_r(z) \subset U$.

(a) *Show every open disk is open.*

(b) *Find a set* A *in* $\mathbb{C}$ *which is* **not** *open and such that its complement* $A^c = \mathbb{C}\backslash A$ *is also* **not** *open. That is,* A *is neither open nor closed.*

(c) *Find the* **diameter** *of*

$$\bigcup_{0<t<\pi/2} D_{e^t}(e^t e^{it}).$$

Remember the diameter of a set S *in* $\mathbb{C}$ *is defined to be*

$$\text{diam}(S) = \sup\{|z_1 - z_2| : z_1, z_s \in S\}.$$

Hint(s)/Suggestion(s): Work in the identification with $\mathbb{R}^2$. *Try to draw a picture of the domain using mathematical software. Parameterize the boundary of each disk for* $0 \le t \le \pi/2$ *by*

$$\gamma(\theta) = e^t[(\cos t, \sin t) + (\cos\theta, \sin\theta)].$$

Find an **envelope curve**

$$\eta(t) = e^t[(\cos t, \sin t) + (\cos\phi(t), \sin\phi(t))]$$

for $0 \le t \le \pi/2$ *and an appropriate function* $\phi : [0, \pi/2] \to [0, \pi/2]$ *for which*

$$\eta'(t) \quad \text{and} \quad \gamma'(t) \qquad \textit{are parallel for each } t.$$

(d) *Stein defines the* **closure** *of a set* $A \subset \mathbb{C}$ *to be*

$$\overline{A} = A \cup A^*$$

where A^* *is the set of* **limit points** *of* A*, i.e., the points* $z \in A$ *for which there is a sequence* $\{z_n\}_{n=1}^{\infty} \subset A\backslash\{z\}$ *with*

$$\lim_{n\to\infty} z_n = z.$$

Show there is always a closed set containing any set A *and the closure* $\overline{A}$ *is also the intersection of all closed sets containing* A*:*

$$\overline{A} = \bigcap_{U^c \supset A,\ U \textit{ open}} U^c.$$

Problem 2 *(S&S Exercise 1.5) Given an open subset* U *in* $\mathbb{C}$*, we say* U *is* **connected** *if whenever* U_1 *and* U_2 *are open subsets of* $\mathbb{C}$ *with* $U = U_1 \cup U_2$*, then one of the following must hold*

$$U_1 \cap U_2 \ne \phi,\ U_1 = \phi,\ \textit{or } U_2 = \phi.$$

Given an open subset U *in* $\mathbb{C}$*, we say* U *is* **path connected** *if whenever* z_1 *and* z_2 *are points in* U*, there exists a continuous function* $\gamma : [0,1] \to U$ *such that* $\gamma(0) = z_1$ *and* $\gamma(1) = z_2$*. In this case,* γ *is called a path connecting* z_1 *to* z_2 *in* U*.*

Show that if U *is path connected, then* U *is connected. Hint(s): Assume by way of contradiction that* $U = U_1 \cup U_2$ *for open sets* U_1 *and* U_2 *with*

$$U_1 \cap U_2 = \phi,\ U_1 \ne \phi,\ \textit{or } U_2 \ne \phi.$$

Take points $z_j \in U_j$ *for* $j = 1, 2$ *and consider*

$$\sup\{T \in [0,1] : \gamma(t) \in U_1 \textit{ for } 0 \le t < T\}.$$

Here sup*, or the* **supremum** *of a set of real numbers, means the "least upper bound."*

Problem 3 *(S&S Exercise 1.5) Show that if U is an open connected subset of $\mathbb{C}$, then U is path connected. Hint(s): Fix a point $z_0 \in U$. Let U_1 be the collection of all points which can be connected to z_0 by a path in U. Show U_1 is an open set. Show U_1 is also a closed set.*

Problem 4 *(general connected sets) Stein defines on page 7 what it means for an open subset of $\mathbb{C}$ to be connected and what it means for a closed subset of $\mathbb{C}$ to be connected. Any set C is* **connected** *if the following holds*

> *If U_1 and U_2 are open subsets of $\mathbb{C}$ and $C \subset U_1 \cup U_2$, then one of the following must hold*
>
> $$U_1 \cap U_2 \neq \phi,\ U_1 \cap C = \phi,\ \text{or } U_2 \cap C = \phi.$$

(a) *Show that when C is a closed connected set according to the general definition above, then C is a closed connected set according to Stein's definition.*

(b) *Show that when C is a closed connected set according to Stein's definition, then C is connected according to the general definition above.*

(c) *Give an example of a closed connected subset of $\mathbb{C}$ which is* **not** *path connected.*

Problem 5 *(S&S Exercise 1.6) Given any open set $U \subset \mathbb{C}$ and a point $z_0 \in U$, we say an open set V is the* **component of U containing z_0** *if V is the largest connected subset of U with $z_0 \in V$.*

(a) *Show that the component V of an open set U containing a point z_0 is the set of all points $z \in \mathbb{C}$ that can be connected to z_0 by a path in U.*

(b) *Show that if V_1 is the component of U containing a point z_1 and V_2 is the component of U containing a point z_2, then either*

$$V_1 \cap V_2 = \phi \qquad \text{or} \qquad V_1 = V_2.$$

Thus, the components of U partition U.

(c) *Show that if U^c is a compact set, then U has exactly one unbounded component.*

I'm going to attempt to standardize some notation or at least make some convenient notation semi-standard for this course. Recall the identification between complex numbers and points in the real Euclidean plane according to which $z = x + iy$ is identified with $(x, y) \in \mathbb{R}^2$ and sets $A \subset \mathbb{C}$ and $S \subset \mathbb{R}^2$ are identified by

$$A = \{z = x + iy \in \mathbb{C} : (x, y) \in S\} \qquad \text{or} \qquad S = \{(x, y) \in \mathbb{R}^2 : z = x + iy \in A\}.$$

Going along with this identification is an identification of functions $f : A \to \mathbb{C}$ and mappings $\Phi : S \to \mathbb{R}^2$ with the real and imaginary parts of f giving rise to component functions of the mapping Φ according to

$$f = u + iv \qquad \text{where} \qquad \begin{cases} u : S \to \mathbb{R} \quad \text{by} \quad u(x, y) = \operatorname{Re} f(x + iy), \textit{and} \\ v : S \to \mathbb{R} \quad \text{by} \quad v(x, y) = \operatorname{Im} f(x + iy). \end{cases}$$

We are especially interested in the differentiability properties of functions f, u, v, and mappings Φ and their relations. This will usually be discussed with reference to particular **open** subsets of $\mathbb{C}$ and $\mathbb{R}^2$ identified as above. I wish to standardize notation for this. Namely, in addition to the identifications above, we will consider $f : \Omega \to \mathbb{C}$ with Ω an open subset of $\mathbb{C}$ and generally assume the associated mapping is $\Phi : U \to \mathbb{R}^2$ is identified with f and $U = \{(x, y) : z = x + iy \in \Omega\}$ is open in $\mathbb{R}^2$.

In this context we will use the the continuity classes of real valued functions as follows:

(a) $C^0(S)$ the collection of all continuous real valued functions on (any set) $S \subset \mathbb{R}^2$.

(b) $C^k(U)$ the collection of all real valued functions with domain an open set $U \subset \mathbb{R}^2$ and having partial derivatives of orders $1, 2, \ldots, k$ in $C^0(U)$.

(c) $C^k(U \to \mathbb{R}^2)$ the collection of all mappings with component functions in $C^k(U)$.

We may also employ minor variations of these notations which (hopefully) will be self-explanatory when they appear.

Problem 6 *(S&S Exercise 1.7) Given a fixed* $w \in D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$*, consider* $f : \overline{D_1(0)} \to \mathbb{C}$ *by*

$$f(z) = \frac{w - z}{1 - \bar{w}z}.$$

(a) *Show that if* $\zeta, z \in D_1(0)$*, then*

$$\left|\frac{\zeta - z}{1 - \bar{\zeta}z}\right| < 1.$$

(b) *Show that if* $\zeta, z \in \mathbb{C}$ *with* $\bar{\zeta}z \neq 1$ *and either* $|\zeta| = 1$ *or* $|z| = 1$*, then*

$$\left|\frac{\zeta - z}{1 - \bar{\zeta}z}\right| = 1.$$

(c) *Show* $f : D_1(0) \to D_1(0)$ *is one-to-one and onto.*

(d) *Show* f *is holomorphic.*

(e) *Show* $f(w) = 0$ *and* $f(0) = w$*.*

(f) *Show* $f : \partial D_1(0) \to \partial D_1(0)$ *is one-to-one and onto.*

The expression

$$\frac{\zeta - z}{1 - \bar{\zeta}z}$$

is called a Blaschke factor. The function f *above given by a single Blaschke factor is an example of a Möbius transformation. The function given by* $e^{i\phi}f$ *obtained by composing a rotation with* f *is also a Möbius transformation.*

Problem 7 *(S&S Exercise 1.8, complex chain rules) Given* $f : \Omega \to W$ *where* Ω *and* W *are open subsets of* $\mathbb{C}$ *and* $g : W \to \mathbb{C}$*, show the following: If* f *and* g *are (complex) differentiable, then*

(a) $g \circ f : \Omega \to \mathbb{C}$ *is differentiable.*

(b) $(g \circ f)' = (g' \circ f)f'$*.*

(c)

$$\frac{\partial(g \circ f)}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}. \qquad \text{and} \qquad \frac{\partial(g \circ f)}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

Problem 8 *(S&S Exercise 1.9) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open subset Ω in $\mathbb{C}$ for which the polar coordinates map $\Psi : U \to V$ by $\Psi(r,\theta) = (r\cos\theta, r\sin\theta)$ is a diffeomorphism[1] where $V = \{(x,y) \in \mathbb{R}^2 : z = x + iy \in \Omega\}$.*

(a) *Let $\xi : U \to \mathbb{R}$ and $\eta : U \to \mathbb{R}$ by*

$$\xi = \xi(r,\theta) = \mathrm{Re}[f \circ \psi^{-1}(r,\theta)] \qquad \text{and} \qquad \eta = \eta(r,\theta) = \mathrm{Im}[f \circ \psi^{-1}(r,\theta)].$$

Show that the Cauchy-Riemann equations for $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ are equivalent to

$$\frac{\partial \xi}{\partial r} = \frac{1}{r}\frac{\partial \eta}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial \xi}{\partial \theta} = -\frac{\partial \eta}{\partial r}.$$

These are called the Cauchy-Riemann equations in polar coordinates.

(b) *Apply part (a) to the functions $\xi : U \to \mathbb{R}$ and $\eta : U \to \mathbb{R}$ by*

$$\xi(r,\theta) = \log r \qquad \text{and} \qquad \eta(r,\theta) = \theta$$

to conclude that the function $f : \Omega \to \mathbb{C}$ by

$$f(z) = \log|z| + i\,\mathrm{Arg}(z)$$

is holomorphic. Such a function is called a branch of the complex logarithm. (You are intended to use Stein's Theorem 2.4 for this problem.)

(c) *Compute*

$$e^{f(z)}$$

where f is a branch of the complex logarithm.

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[1] That is a one-to-one, onto, and continuous function with a continuous inverse.

Problem 9 *(S&S Exercise 1.10) Given real valued functions* $u, v \in C^2(U)$ *where* U *is an open set in* $\mathbb{R}^2$*, consider* $f : \Omega \to \mathbb{C}$ *by* $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ *where* $\Omega = \{z = x + iy \in \mathbb{C} : (x,y) \in U\}$.

(a) *Show*

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{1}{4}\Delta f$$

where $\Delta : C^2(\Omega \to \mathbb{C}) \to C^0(\Omega \to \mathbb{C})$ *by*

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

is the extension of the usual Laplace operator to complex valued functions. Note: It is not required that f *is complex differentiable here.*

(b) *Give an example of a function* f *to which part (a) applies but which is not holomorphic.*

Problem 10 *(S&S Exercise 1.11) Use the previous exercise to show that if* $f : \Omega \to \mathbb{C}$ *is harmonic and* $f = u + iv$ *with the usual identifications so that* $u, v \in C^2(U)$ *with* $U = \{(x,y) \in \mathbb{R}^2 : z = x + iy \in \Omega\}$*, then* u *and* v *satisfy*

$$\Delta u = 0 \qquad \text{and} \qquad \Delta v = 0.$$

Note: A function $\phi \in C^2(\Omega)$ *where* Ω *is an open subset of* $\mathbb{R}^2$ *is called* **harmonic** *if* $\Delta\phi = 0$*. Note: We may not have yet proved that the real and imaginary parts* u *and* v *of a holomorphic function are twice continuously differentiable, but this is true, and we will prove it.*