Assignment 11: Complex Analysis
Due Tuesday April 28, 2022

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Problem 1 *(Area Deficiency) For* $r > 0$ *consider the sphere*

$$\partial B_r(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = r\}$$

parameterized (in spherical coordinates) by

$$X(\phi, \theta) = r(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

More specificially, consider the spherical cap $\mathcal{S}$ *centered at* $(0, 0, -r)$ *corresponding to* $(\phi, \theta) \in [\pi - \psi, \pi] \times [0, 2\pi]$ *for some* ψ *with* $0 < \psi < \pi/2$*. Notice this is the cap subtended by a cone with vertex at* $\mathbf{0} \in \mathbb{R}^3$*, vertical axis, and cone angle* 2ψ*.*

(a) *Compute the area* A *of* $\mathcal{S}$ *as a function of* r *and* ψ*.*

(b) $\mathcal{S}$ *is also the intersection of a ball* $\overline{B_a(\mathbf{x}_0)}$ *with center* $\mathbf{x}_0 = (0, 0, -r)$ *with* $\partial B_r(\mathbf{0})$*. Find the relation between the (ambient) radius* a *and (half) cone angle* ψ*.*

(c) $\mathcal{S}$ *may be referred to as a* **geodesic disk** *in* $\partial B_r(\mathbf{0})$ *with center* $\mathbf{x}_0 = (0, 0, -r)$*. Find the distance from* $\mathbf{x}_0$ *to a point* $\mathbf{x} \in \partial B_a(\mathbf{x}_0)$ *measured within* $\mathcal{S}$ *defined by*

$$\begin{aligned}\rho &= \text{dist}(\mathbf{x}_0, \mathbf{x}) \\ &= \min\left\{\int_0^1 |\alpha'(t)|\, dt : \alpha \in C^1([0,1] \to \mathcal{S}) \text{ with } \alpha(0) = \mathbf{x}_0,\ \alpha(1) = \mathbf{x}\right\}.\end{aligned}$$

ρ *is called the geodesic radius (or simply the radius) of* $\mathcal{S}$ *as a geodesic disk.*

(d) *The set $\partial B_a(\mathbf{x}_0) \cap \partial B_r(\mathbf{0})$, which is a circle of course, is called the boundary of $\mathcal{S}$ and is denoted by $\partial\mathcal{S}$, though one should note that the meaning of "partial" or "∂" in $\partial\mathcal{S}$ is somewhat different from the use of ∂ to denote the boundary of subsets of $\mathbb{R}^3$. The ∂ in $\partial\mathcal{S}$ denotes the topological boundary of $\mathcal{S}$ as a surface, or equivalently as a metric subspace of $\partial B_r(\mathbf{0})$—the sphere itself being considered a metric subspace of $\mathbb{R}^3$.*

(i) *What is $\partial\mathcal{S}$ as a subset of $\mathbb{R}^3$?*

(ii) *What is the boundary $\partial B_r(\mathbf{0})$ as a surface?*

(iii) *What is the radius of the boundary circle $\partial\mathcal{S}$?*

(e) *Compute the Besicovich area density*

$$\lim_{a \searrow 0} \frac{A}{\pi a^2}.$$

(f) *Compute*

$$\lim_{\rho \searrow 0} \frac{A}{\pi \rho^2}.$$

(g) *Compute the* **Gauss area deficiency**

$$\lim_{\rho \searrow 0} 12 \frac{\pi\rho^2 - A}{\pi\rho^4}.$$

(h) *Compute the area deficiency*

$$\lim_{a \searrow 0} 12 \frac{\pi a^2 - A}{\pi a^4}.$$

Do your answers to parts **(g)** *and* **(h)** *make sense?*