The Abel Limit Theorem
And Boundary Behavior of Complex Power Series

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Stein and Shakarchi have a simplified version of the Abel limit theorem in the exercises with a hint to use Stein's exercise on partial summation. I had worked through a somewhat more general version of Abel's theorem as presented in Ahlfors in my notes from 2018, but I think I can simplify that presentation a little bit. Any way one does it, the manipulations to prove this result are a bit complicated, so I'm going to present several different approaches and prove both the simplified and the more general version.

1 Boundary Behavior

Recall that the general theorem on convergence of complex power series gives a disk of convergence with absolute convergence inside the disk and divergence outside the disk. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \cdots \tag{1}$$

has Hadamard radius

$$R = \frac{1}{\lim_{n\to\infty} 1/n^{1/n}} = 1$$

since

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} \sim \lim_{n\to\infty} \frac{1}{n} = 0.$$

The theorem tells us nothing, however, about the convergence of the series when $|z| = R$. Furthermore, there are interesting cases where it is relatively easy to determine the convergence or divergence of the series when $|z| = R$, but we know nothing apriori about the relation between the values at the boundary and the values in the interior. For example when $z = 1$ in the series (1) we get the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

of real numbers. This series can easily seen to converge to some number between 1/2 and 1, between 7/12 and 5/6, and so on, but we might like to know what number this is precisely. This is where the Abel limit theorem comes in. It turns out that for this series the function represented on (interior of) the disk $D_1(0)$ is a well-known standard function with a simple formula, tabulated values, and subject to standardized calculation with mathematical software or even a scientific calculator. In particular, for the real number x satisfying $0 < x < 1$, the value

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \tag{2}$$

is **known**. In fact, f is known to be well-defined and continuous for $x \in (-1, \infty)$ with well-defined (and known) value $f(1)$. The Abel limit theorem tells us, for example that the value of the alternating harmonic series is $f(1)$.

Theorem 1 (Abel limit theorem version 1) *If the complex power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R = 1$ representing a complex differentiable function $f : D_1(0) \to \mathbb{C}$ on the open unit disk, and

$$\sum_{n=0}^{\infty} a_n = w \in \mathbb{C},$$

then

$$w = \lim_{x \nearrow 1} f(x) = \lim_{x \nearrow 1} \sum_{n=1}^{\infty} a_n x^n.$$

A proof of this theorem is assigned as an exercise in Stein and Shakarchi and is outlined in Problem 4 of Assignment 4 for my complex analysis course Spring 2022.

In the same Assignment one should be able to find, relatively, easily the identity of the function defined in (2) and, consequently, the numerical value of the alternating harmonic series. The proof of Stein and Shakarchi using the "summation by parts" formula is a little slick I find. I'm going to offer another proof, which is also tricky and sort of amounts to the same thing, but may conceivably also be considered somehow more straightforward. I imagine (and hope) so.

2 Proof of Theorem 1

The basic idea is simple: The coefficients a_n in

$$\sum a_n x^n$$

can be replaced with differences of the partial sums of the series $\sum a_n$. To be precise,

$$a_n = S_n - S_{n-1} = \sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j \qquad \text{for } n = 1, 2, 3, \ldots \tag{3}$$

where S_k given by

$$S_k = \sum_{n=0}^{k} a_n$$

is the k-th partial sum of the series $\sum a_n$ as usual.

Before attempting to use this simple observation, let us verify that the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely and thus defines a complex differentiable function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

on the open unit disk. One simple way to do this is to consider the Hadamard radius

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

By the main theorem on the convergence of (complex) power series, we know the formal series diverges for $|z| > R$. In particular, since the series is known to converge for $z_1 = 1$, we know $1 \leq R$, and we are done. Note that the actual disk of convergence determined by R may be bigger than the unit disk, but the interesting applications of the theorem are when $R = 1$.

Note also that one approach to verifying this convergence which does **not** work in such a straightforward manner is **comparison** as is familiar with real series. The series of absolute values

$$\sum_{n=0}^{\infty} |a_n|\,|z|^n$$

is indeed a real series with non-negative terms and $|a_n|\,|z|^n < |a_n|$ for $|z| < 1$, but we are **not** assuming the series

$$\sum_{n=0}^{\infty} |a_n|$$

converges, but only that

$$\sum_{n=0}^{\infty} a_n \qquad \text{converges.}$$

In fact, the interesting cases (or at least many interesting cases) are when $\sum a_n$ converges "conditionally" rather than absolutely.

With this in mind, we consider the quantity

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n = f(x) - w \tag{4}$$

which we wish to show is small when $|x - 1| = 1 - x$ is small. This is our basic objective.

From here on, things are a bit tricky. We argue in two steps. The first step is to express the value in (4) in a basically different form as a different (convergent) series, or at least involving a different series. To this end, consider a partial sum limiting to the quantity in (4):

$$\sum_{n=0}^{k} a_n x^n - w. \tag{5}$$

Notice the first coefficient is an exception not subject to (3), so we we write (5) as

$$a_0 - w + \sum_{n=1}^{k} a_n x^n = a_0 - w + \sum_{n=1}^{k} (S_n - S_{n-1}) x^n. \tag{6}$$

Now, we write out the summation:

$$a_0 - w + \sum_{n=1}^{k} a_n x^n = S_0 - w \sum_{n=1}^{k} x^n (S_n - S_{n-1}) \tag{7}$$

$$= S_0 - w \quad + \quad x(S_1 - S_0) + x^2(S_2 - S_1) + x^3(S_3 - S_2) + \cdots + x^k(S_k - S_{k-1}).$$

Notice that we have written the sum at the end with suggestive spacing so that a, more or less, obvious regrouping of these funitely many terms gives

$$\begin{aligned} a_0 - w + \sum_{n=1}^{k} a_n x^n &= S_0(1-x) - w + xS_1(1-x) \\ &\quad + x^2 S_2(1-x) \\ &\quad + x^3 S_3(1-x) + \cdots + x^k S_k \\ &= S_0(1-x) - w + (1-x)\sum_{n=1}^{k-1} x^n S_n + x^k S_k \\ &= (1-x)\sum_{n=0}^{k-1} x^n S_n - w + x^k S_k. \end{aligned}$$

This is close to what we want, but it turns out it is not quite good enough. In particular, the last term $x^k S_k$ looks like it might limit to w as $k \to \infty$ and $x \nearrow 1$, but what we really need is to fix x and let $k \nearrow \infty$, and in this case (with $x < 1$) we clearly don't get that. So, let's go back to (7) where we wrote out the sum and do something tricky. Notice that each difference $S_n - S_{n-1}$ can also be written as $S_n - w - (S_n - w)$. If we do this and then do the same regrouping, we get

$$\begin{aligned} a_0 - w + \sum_{n=1}^{k} a_n x^n &= S_0 - w + \sum_{n=1}^{k} x^n [S_n - w - (S_{n-1} - w)] \\ &= (1-x)(S_0 - w) + x(S_1 - w)(1-x) + x^2(S_2 - w)(1-x) \\ &\quad + \cdots x^{k-1}(S_{k-1} - w)(1-x) + x^k(S_k - w) \\ &= (1-x)\sum_{n=0}^{k-1} x^n (S_n - w) + x^k(S_k - w). \end{aligned}$$

This is much better. First of all the left over terms at the end can be made small independent of N: We know $S_k \to w$, so for any $\epsilon > 0$ there is some N for which $k > N$ implies

$$|x^k(S_k - w)| \le |S_k - w| < \epsilon. \tag{8}$$

In particular,

$$\lim_{k\to\infty} x^k(S_k - w) = 0.$$

Now, taking this term to the other side we have

$$(1-x)\sum_{n=0}^{k-1} x^n(S_n - w) = \sum_{n=0}^{k} a_k x^k - w - x^k(S_k - w).$$

Taking the limit as $k \to \infty$ therefore, we get that the product with the sum on the left has a limit and

$$\sum_{n=0}^{k} a_k x^k - w = (1-x)\sum_{n=0}^{\infty} x^n(S_n - w). \tag{9}$$

This is what we meant be expressing the quantity in (4) in terms of a different (convergent) series. This completes the first step.

For the second step, we use the fact that the series on the right in (9) has some nice properties. To be precise, we can write

$$\begin{aligned}\sum_{n=0}^{k} a_k x^k - w &= (1-x)\sum_{n=0}^{\infty} x^n(S_n - w)\\ &= (1-x)\sum_{n=0}^{N} x^n(S_n - w) + (1-x)\sum_{n=N+1}^{\infty} x^n(S_n - w).\end{aligned} \tag{10}$$

Now we let $\epsilon > 0$ and take N so that $n > N$ implies

$$|S_n - w| < \frac{\epsilon}{2}.$$

In particular, this means the second summation in (10) can be estimated by

$$\begin{aligned}\left|(1-x)\sum_{n=N+1}^{\infty}x^n(S_n-w)\right| &= (1-x)\left|\sum_{n=N+1}^{\infty}x^n(S_n-w)\right| \\ &\le \frac{\epsilon}{2}(1-x)\sum_{n=N+1}^{\infty}x^n \\ &\le \frac{\epsilon}{2}(1-x)\sum_{n=0}^{\infty}x^n \\ &= \frac{\epsilon}{2}.\end{aligned}$$

Now, with N fixed so that this holds, we consider the first summation in (10):

$$\left|(1-x)\sum_{n=0}^{N}x^n(S_n-w)\right| = (1-x)\left|\sum_{n=0}^{N}x^n(S_n-w)\right| \le (1-x)\sum_{n=0}^{N}|S_n-w|.$$

Since

$$M=\sum_{n=0}^{N}|S_n-w|$$

is just a fixed non-negative number, we can take

$$1-x<\frac{\epsilon}{2(M+1)}$$

and conclude that this implies

$$\left|\sum_{n=0}^{k}a_kx^k-w\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \qquad \square$$

3 Stolz Angle

4 Another Example: Leibniz Series

The complex arctangent function, or a branch of it, may be defined on the open unit disk according to the relation

$$\frac{d}{dz}\tan^{-1}z=\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=\sum_{n=0}^{\infty}(-1)^nz^{2n}. \tag{11}$$

To choose the branch we need to specify $\tan^{-1}(0) = 0$, and technically, we may start directly with the term by term integrated series:

$$\tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}. \tag{12}$$

Exercise 1 *Define a function $f : D_1(0) \to \mathbb{C}$ by the series in (12). Use the differentiation theorem for complex power series to conclude*

$$f'(x) = \frac{1}{1+z^2}.$$

Exercise 2 *Given the complex function f from the previous exercise, show the real function $\tan^{-1} : \mathbb{R} \to \mathbb{R}$ with $\tan^{-1}(0) = 0$ satisfies*

$$f(x) = \tan^{-1}(x) \quad \textit{for } x \in (-1, 1) \subset \mathbb{R}.$$

Hint, restrict f to the real line to obtain a real valued function g of a real variable on $(-1, 1)$; show the real derivative g' agrees with the complex derivative f' along the interval $(-1, 1)$.

Taking $z = 1$ in the formal power series given in (12) we see the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

which is convergent as an alternating series of real terms with the individual terms tending to 0. The Abel limit theorem tells us that the value may be obtained as the limit

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \lim_{x \nearrow 1} \tan^{-1}(x) = \tan^{-1}(1) = \frac{\pi}{4}.$$

5 Boundary Point for Arcsine

I am going to attempt here to justify carefully the following formulas

$$\frac{d}{dz} \sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\ 2^{2k}}\ z^{2k}, \tag{13}$$

$$\sin^{-1}(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\ 2^{2k}\ (2k+1)}\ z^{2k+1}, \tag{14}$$

and

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 \, 2^{2k} \, (2k+1)} = 1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \cdots = \sin^{-1}(1) = \frac{\pi}{2}. \qquad (15)$$

A central aspect of what we wish to establish concerning the first two formulas involves the fact that these are formulas valid for a complex variable z. Thus, precise domains (and codomains) in the complex plane need to be specified, and we need to be careful about the functions we use. In particular, the identity and nature of the square root appearing in (13) should be discussed. In fact, let us recall that the formula

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \text{for} \qquad |x| < 1$$

is typically obtained as follows: We know $\sin(\sin^{-1} x) = x$ on the interval $-1 \le x \le 1$, and we can differentiate (using the chain rule) in the interior of the interval to obtain

$$\cos(\sin^{-1} x) \, \frac{d}{dx} \sin^{-1} x = 1.$$

Furtherfore, we have the identity $\cos^2\theta + \sin^2\theta = 1$. In particular, since $\sin^{-1} : [-1,1] \to [-\pi/2, \pi/2]$ and $\cos : [-\pi/2, \pi/2] \to [0,1]$, we know

$$\cos^2(\sin^{-1} x) = 1 - \sin^2(\sin^{-1} x) = 1 - x^2.$$

Given that $\cos\theta \ge 0$ for $-\pi/2 \le \theta \le \pi/2$, there is no ambiguity in taking the positive square root and writing

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}.$$

Much of this is valid for the complex arcsine. In particular, starting from the series definitions

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \quad \text{and} \quad \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1},$$

we obtain three entire holomorphic functions whose mapping properties can be determined and for which various identities hold. Among these properties we find that $\sin : \mathbb{C} \to \mathbb{C}$ covers the complex plane exactly once on the set/strip

$$\Sigma = \{x + iy \in \mathbb{C} : |x| < \pi/2\} \cup \{\pi/2 + iy : y \ge 0\} \cup \{-\pi/2 + iy : y \le 0\}$$

with the points $z = \pm\pi/2 \in \mathbb{R}$ mapping to branch points at $w = \pm 1$. In particular, sin is invertible on this set defining an inverse $\sin^{-1} : \mathbb{C} \to \Sigma$ which is differentiable on the unit disk $D_1(0)$ satisfying

$$\sin(\sin^{-1}(w)) = w$$

so that

$$\cos(\sin^{-1}(w))\,\frac{d}{dw}\sin^{-1}(w) = 1 \tag{16}$$

just as in the real case. The restricted mapping $z = \sin^{-1} : D_1(0) \to \mathbb{C}$ under consideration here is illustrated in Figure 1. The relations

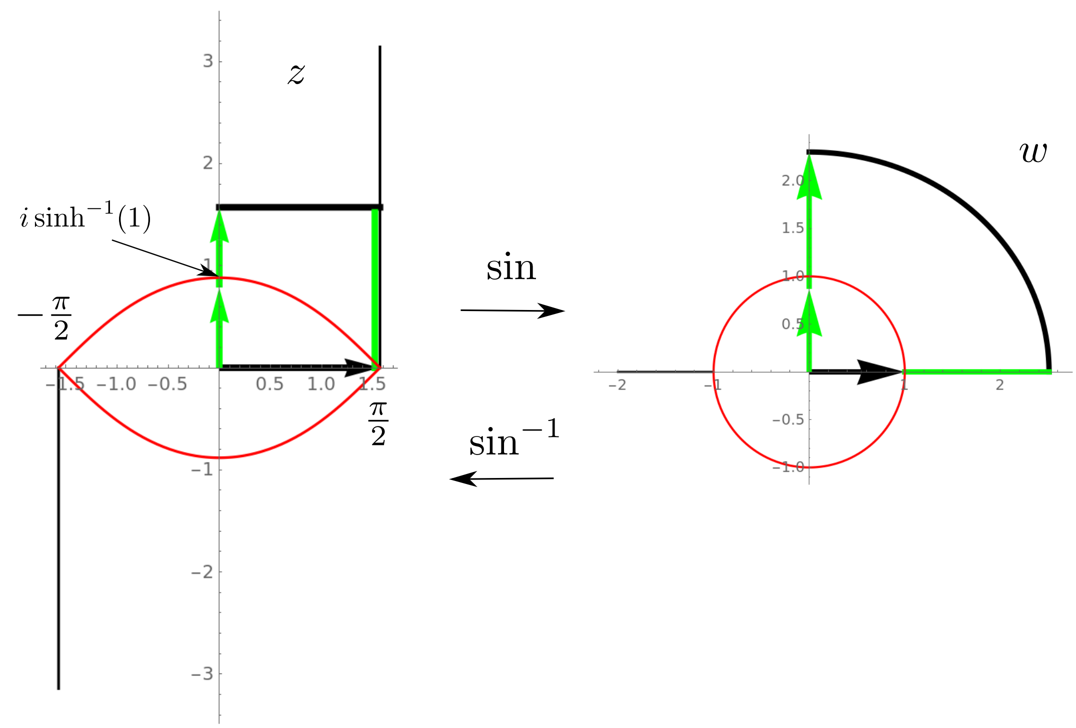

Figure 1: The complex sine function and arcsine function.

$$\cos(a + bi) = \cos a \cosh b - i \sin a \sinh b$$
$$\sin(a + bi) = \sin a \cosh b + i \cos a \sinh b$$

for $a, b \in \mathbb{R}$ can be helpful in understanding the mapping properties of the complex trigonometric functions like those illustrated in Figure 1. In particular, if one looks for the inverse image of the unit disk, then one may wish to consider

$$\{a + bi \in \Sigma : |\sin(a + bi)| = 1\}.$$

That is,

$$\sin^2 a \cosh^2 b + \cos^2 a \sinh^2 b = \cosh^2 b - \cos^2 a = 1$$

or $\sinh^2 b = \cos^2 a$ so that

$$b = \pm \sinh^{-1}(\cos a).$$

Let

$$\Omega = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \sinh^{-1}(\cos \operatorname{Re}(z)),\ -\pi/2 < \operatorname{Re}(z) < \pi/2\}$$

be the lens shaped image codomain of arcsine on the unit disk.

We also have the relation

$$\cos^2 z + \sin^2 z = 1$$

in the complex case so that

$$\cos^2(\sin^{-1}(w)) = 1 - \sin^2(\sin^{-1}(w)) = 1 - w^2 \tag{17}$$

is valid. In order to take the square root, we should consider the image of the complex cosine on Ω. For this we may also use the identities

$$\cos(z) = \cos(-z) = \sin(z + \pi/2) \qquad \text{and} \qquad \sin(z) = -\sin(-z).$$

For example, to determine the image of the portion

$$A = \{z : 0 \le \operatorname{Re}(z) < \sinh^{-1}(\cos \operatorname{Re}(z)),\ 0 \le \operatorname{Re}(z) < \pi/2\}$$

of Ω in the first quadrant we can write

$$\begin{aligned}
&\{\cos(z) : z \in A\} \\
&\qquad = \{\cos(z) : 0 \le \operatorname{Im}(z) < \sinh^{-1}(\cos \operatorname{Re}(z)),\, 0 \le \operatorname{Re}(z) < \pi/2\} \\
&\qquad = \{\cos(z) : \sinh^{-1}(\cos \operatorname{Re}(z)) < \operatorname{Im}(z) \le 0,\, -\pi/2 < \operatorname{Re}(z) < 0\} \\
&\qquad = \{\sin(z + \pi/2) : -\sinh^{-1}(\cos \operatorname{Re}(z)) < \operatorname{Im}(z) \le 0,\, -\pi/2 < \operatorname{Re}(z) < 0\} \\
&\qquad = \{\sin(z) : -\sinh^{-1}(\pi/2 - \cos \operatorname{Re}(z)) < \operatorname{Im}(z) \le 0,\, 0 < \operatorname{Re}(z) \le \pi/2\}.
\end{aligned}$$

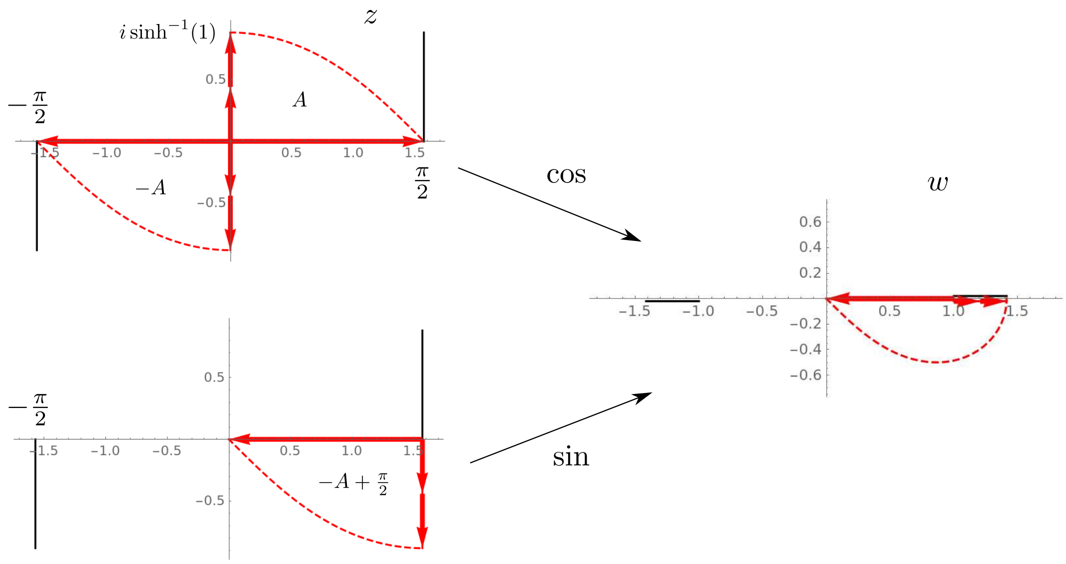

Figure 2: Using the complex sine function to find the image of a set A under the complex cosine.

Consequently, we can see the image of the region $A \subset \Omega$ is in the fourth quadrant as indicated in Figure 2. Note, first of all that our reasoning also shows that the region

$$-A = \{-z : z \in A\} = \{z : -\sinh^{-1}(\cos \mathrm{Re}(z)) < \mathrm{Im}(z) < 0,\ -\pi/2 < \mathrm{Re}(z) \leq 0\}$$

has precisely the same image under cosine. This means, in particular, that the image of Ω under cosine is not one-to-one. It makes sense therefore to represent this image in some kind of Riemann surface as we have done in Figure 3 in which the image of $-A$ under cosine is represented in a second sheet. The same approach applies to the conjugate regions

$$\overline{A} = \{\bar{z} : z \in A\} = \{z : \sinh^{-1}(\cos \mathrm{Re}(z)) < \mathrm{Im}(z) \leq 0,\ 0 \leq \mathrm{Re}(z) < \pi/2\}$$

and $-\overline{A}$ of Ω in the second and fourth quadrants which both have image conjugate to $\cos(A)$ as indicated in Figure 4. Note that the Riemann surface in Figure 4 has the branch cut extending from $w = 1$ in the direction of the negative real axis. In particular, in this presentation there is a smooth transition along $\mathrm{Re}\, w > 1$ on both sheets. The teardrop domain of interest may seem somewhat "exotic," but it should be emphasized that all associated quantities associated with it should seem reasonable, if not straightforward, to calculate.

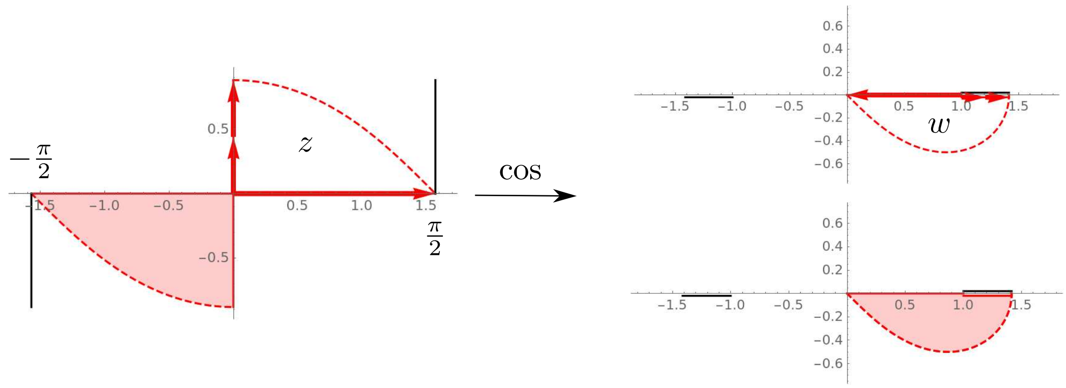

Figure 3: The images of sets A and $-A$ under the complex cosine in a Riemann surface.

Exercise 3 *Note that the upper boundary curve of the lens region is given parametrically as a graph by*

$$\gamma(t) = t + i\sinh^{-1}(\cos t) \qquad \text{for} \qquad -\pi/2 \leq t \leq \pi/2,$$

and consequently, the upper boundary of the teardrop region, according to the formula for cosine, should be given parametrically by

$$\eta(t) = \cos t\cosh(\sinh^{-1}(\cos t)) + i\sin t\sinh(\sinh^{-1}(\cos t)) = \cos t\cosh(\sinh^{-1}(\cos t)) + i\cos t\sin t$$

Write $x = \cos t\cosh(\sinh^{-1}(\cos t))$ *and* $y = \cos t\sin t$*. Eliminate* t *to find the upper boundary of the teardrop region non-parametrically as* $y = y(x)$*. Determine the rightmost boundary point of the teardrop region.*

Solution: Since $\cosh^2 z = 1 + \sinh^2 z$, we can write the first relation as

$$x = \cos t\sqrt{1 + \cos^2 t}.$$

Squaring both sides, we get a quartic (really biquadratic) equation for $\cos t$:

$$\cos^4 t + \cos^2 t - x^2 = 0.$$

From this we get

$$\cos^2 t = \frac{-1 + \sqrt{1 + 4x^2}}{2} \qquad \text{and} \qquad \cos t = \sqrt{\frac{-1 + \sqrt{1 + 4x^2}}{2}}.$$

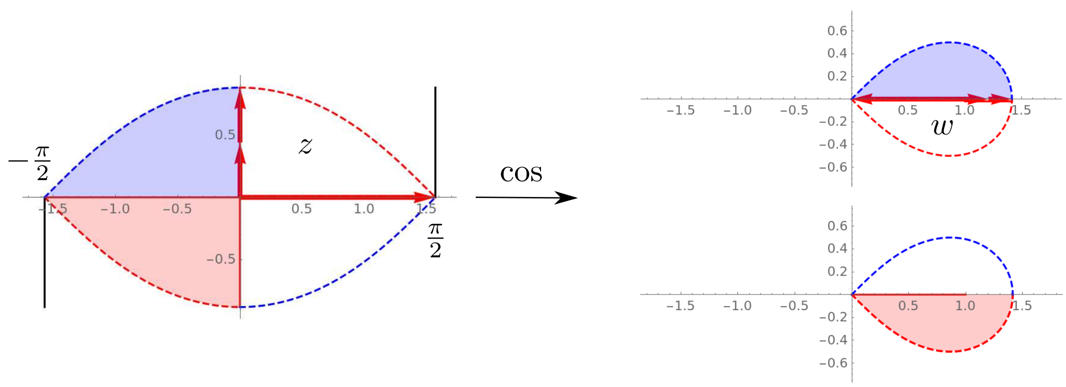

Figure 4: The image of the lens shaped domain Ω under the complex cosine in a Riemann surface.

In this case,

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{\frac{3 - \sqrt{1 + 4x^2}}{2}}.$$

Hence

$$y = \cos t \sin t = \sqrt{\sqrt{1 + 4x^2} - (1 + x^2)}.$$

The rightmost point on $\partial \cos(\Omega)$ is

$$\eta(0) = \cosh(\sinh^{-1}(1)) = \sqrt{1 + \sinh^2(\sinh^{-1}(1))} = \sqrt{2}.$$

As mentioned above, the (double covered) teardrop domain $\cos(\Omega)$ may seem somewhat exotic, but we are also now in a position to recognize it as something else which is relatively simple. Recall that we do have the relation

$$\cos^2(\sin^{-1}(w)) = 1 - w^2.$$

This means that if we apply the quadratic function $f(\zeta) = \zeta^2$ to the teardrop domain we will get precisely the same set obtained by applying $g(w) = 1 - w^2$ to the unit disk $D_1(0)$. This is a rather simple domain: the unit disk is invariant under $w \mapsto -w^2$ or, more properly $D_1(0)$ maps to a double cover of itself. Thus, the image of the teardrop domain under $f(\zeta) = \zeta^2$ is (a double cover) of $D_1(1)$. To get an even better picture of the "exotic" teardrop domain $\cos(\Omega)$, start with two copies/sheets of $D_1(1)$ sewn

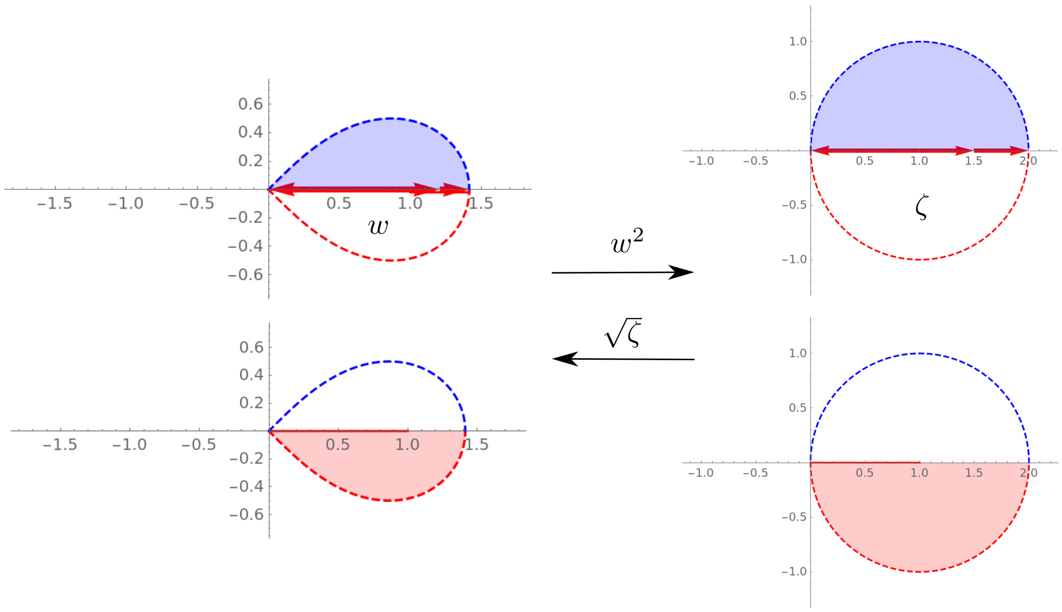

Figure 5: The teardrop domain $\cos(\Omega)$ is actually not so exotic.

together as a Riemann surface along the interval $(0,1) \subset \mathbb{R}$, and then simply apply the principal branch of the square root; see Figure 5.

We have just established the formula

$$\cos(\sin^{-1}(w)) = \sqrt{1-w^2} \tag{18}$$

unambiguously for the complex variable $w \in D_1(0)$ where the square root appearing in (18) is the (complex) principal square root. It was a lot of work, but we have accomplished something. In particular, we can return to (16) and write

$$\frac{d}{dw}\sin^{-1}(w) = \frac{1}{\sqrt{1+w^2}} \qquad \text{for} \qquad |w| < 1$$

as a complex formula where (hopefully) we understand everything we have written. This is our first major objective and establishes the first equality in (13).

5.1 Formal Expansions and Formal Expansion Techniques

Unfortunately, I don't think we have the tools in hand to rigorously verify the second equality in (13). There are various calculations we can make to provide "circumstantial evidence" that (13) is correct, and I am planning to do that in this section. For example, we can consider the generalized binomial expansion of

$$\frac{1}{\sqrt{1+w^2}} = (1+w^2)^{-1/2}.$$

Since such things may be unfamiliar, I will give some details. The **binomial expansion formula** is usually written as

$$(a+b)^k = \sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array} \right) a^{k-n}\, b^n$$

for an integer $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ where

$$\left(\begin{array}{c} k \\ n \end{array} \right) = \frac{k!}{(k-n)!n!} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

is the **binomial coefficient** or **combination** of k things taken n at a time with a product of n terms in both the numerator and the denominator of the last expression. This can be verified using induction. We can also write this formula as

$$(a+b)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}\, a^{k-n}\, b^n.$$

Notice that I'm now letting n run from $n = 0$ to ∞ so this is formally a series. When k is an integer, however all the coefficients

$$\frac{k(k-1)\cdots(k-n+1)}{n!}$$

with $n > k$ contain a factor $k - k = 0$ in the numerator and thus vanish. This opens the possibility to consider the expression

$$\sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, a^{p-n}\, b^n$$

where p is a **non-integer power**.[1]

[1] Apparently, Isaac Newton was the first person to think of doing this.

Theorem 2 *The series*

$$\sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\, z^n$$

converges absolutely for $z \in D_1(0)$ *to the function* $f(z) = (1+z)^p$ *for any power* p.

I vaguely remember proving this in 2018, but I don't know if the proof is in the notes. I'm not going to prove it here. I think the basic tool to prove Newton's theorem is Taylor series expansion for holomorphic functions—which we definitely want to prove, but we haven't proved yet. In particular, for

$$\frac{d}{dw} \sin^{-1}(w) = \frac{1}{\sqrt{1-w^2}} = (1 + (-w^2))^{-1/2}$$

we can certainly calculate all the higher order derivatives at 0 and show they match a certain power series. If we had the complex Taylor expansion theorem, then that series would be the correct one.

Exercise 4 *Compute*

$$\frac{d^n}{dw^n}\left(\frac{1}{\sqrt{1-w^2}}\right)\Big|_{w=0}$$

for $n = 1, 2, 3, \ldots$. *What series do you get?*

If the formal expansion

$$f(w) = \sum_{n=0}^{\infty} \frac{(-1/2)(-3/2))\cdots(-(2n-1)/2)}{n!}\, (-w^2)^n$$

from Newton's theorem has coefficients matching the calculation of the derivatives, then that will be a good sign. Let's simplify the expression for the binomial coefficients. We start with

$$f(w) = \sum_{n=0}^{\infty} \frac{(-1)(-3))\cdots(-(2n-1))}{2^n n!}\, (-1)^n w^{2n}.$$

There is a factor of $(-1)^n$ in the numerator of the coefficient which cancels the factor of $(-1)^n$ from the power. Also, the product of odd numbers

$$1(3)(5)\cdots(2n-1) = \frac{(2n)!}{2(4)(6)\cdots(2n)} = \frac{(2n)!}{2^n n!}.$$

Making this substitution, the series takes the nice "closed" form

$$f(w) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\ (n!)^2}\ w^{2n}. \tag{19}$$

The Hadamard limit

$$\limsup_{n\to\infty} \left(\frac{(2n)!}{2^{2n}\ (n!)^2} \right)^{\frac{1}{2n}} \tag{20}$$

may not be so easy to calculate, but the value is 1 and the radius of convergence is $R = 1$. Perhaps the easiest way to get this is by using the result on the ratio test/limit from Problem 7 of Assignment 3. Notice that if we interpret the series for $f(w)$ as

$$f(w) = \sum_{n=0}^{\infty} a_{2n}\ w^{2n} \qquad \text{with} \qquad a_{2n} = \frac{(2n)!}{2^{2n}\ (n!)^2}, \tag{21}$$

then technically every other term in the power series is zero, so the ratios $|a_{n+1}|/|a_n|$ are not so well behaved. One simple way to get around this is to write

$$f(w) = \sum_{n=0}^{\infty} b_n\ (w^2)^n \qquad \text{with} \qquad b_n = \frac{(2n)!}{2^{2n}\ (n!)^2}.$$

Then we can apply the result of Problem 7 of Assignment 3 because

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = 1.$$

Thus, the series

$$\sum_{n=0}^{\infty} b_n\ \zeta^n$$

converges absolutely for $|\zeta| < 1$, and the same thing is therefore true for the series (21) for $|w| < 1$. We have then that $f : D_1(0) \to \mathbb{C}$ given in (21) is holomorphic.

Exercise 5 (a) *Evaluate the limit in (20) directly. Hint: Take the (real) logarithm of the expression first.*

(b) *Use the ratio limit/test result of Assignment 3 Problem 7 to show the series in Newton's Theorem (Theorem 2) has Hadamard radius 1.*

We have a holomorphic function $f : D_1(0) \to \mathbb{C}$ given by (19) as a series

$$f(w) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\ (n!)^2}\ w^{2n}$$

and we would like to assert that this function is

$$\frac{1}{\sqrt{1-w^2}}.$$

For another piece of circumstantial evidence involving a nice formal series manipulation, let's note that if our assertion is correct, then

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\ (n!)^2}\ \zeta^n = \frac{1}{\sqrt{1-\zeta}},$$

Let us consider formally squaring the series. The nice thing about squaring a series is that you only get finitely many terms to add up for each particular power of ζ:

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n \zeta^n\right)\left(\sum_{m=0}^{\infty} a_m \zeta^m\right) &= \sum_{\ell=0}^{\infty}\left(\sum_{n+m=\ell} a_n a_m\right)\ \zeta^\ell \\ &= a_0^2 + (a_0a_1 + a_1a_0)\zeta + (a_0a_2 + a_1^2 + a_2a_0)\zeta^2 + \cdots .\end{aligned}$$

And if the series on the right converges, then it clearly has the same partial sums as the product of partial sums on the left and hence the same limit. In this case, we're interested in the coefficients

$$\sum_{m=0}^{\ell} \frac{(2(\ell-m))!}{2^{2(\ell-m)}\ ((\ell-m)!)^2} \frac{(2m)!}{2^{2m}\ (m!)^2}.$$

By induction on ℓ

$$\sum_{m=0}^{\ell} \frac{(2(\ell-m))!}{2^{2(\ell-m)}\ ((\ell-m)!)^2} \frac{(2m)!}{2^{2m}\ (m!)^2} = 1 \qquad \text{for} \qquad \ell = 0, 1, 2, \ldots .$$

That is,

$$\left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\ (n!)^2}\ \zeta^n\right)^2 = \frac{1}{1-\zeta} \qquad \text{for} \qquad |\zeta| < 1.$$

Thus we are back in the position where we need to take the (principal) square root again, but we are in trouble because it's not obvious that the values of $[f(w)]^2$ avoid the branch cut along the negative real axis. Thus, we can only conclude that the series for $f(w)$ gives **some square root** of $1/(1-w^2)$.

5.2 Application of Abel's Limit Theorem

Moving on to (14) we can clearly write down the expression we have obtained from formal expansion and termwise integration:

$$g(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\ 2^{2k}\ (2k+1)}\ z^{2k+1}.$$

The radius of convergence is again $R = 1$, and we know that if we differentiate the resulting holomorphic function $g : D_1(0) \to \mathbb{C}$, we get $g' = f$. We would like to conclude

$$\sin^{-1}(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\ 2^{2k}\ (2k+1)}\ z^{2k+1},$$

and if (13) were fully justified we would have this. As it stands, we only have circumstantial evidence. Nevertheless, this equality turns out to be true and Abel's theorem then gives (15).