MATH 4581 Lecture 16, Tuesday October 19, 2021

- Fourier's Theorem ((Chapter 3 Haberman)

Coming Soon: The Wave Equation

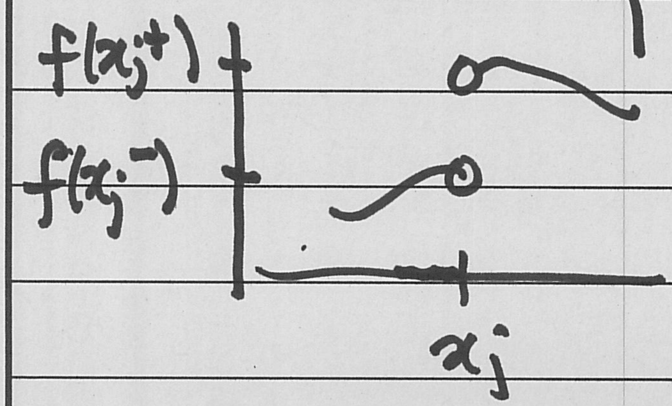
$$u_{tt} = \sigma^2 u_{xx} \quad (n=1)$$

[redacted]

Pointwise Convergence

Fourier's Theorem: If $f : I \to \mathbb{R}$ is piecewise smooth with respect to the countable $\{x_j\}_{j=1}^{\infty}$ of the interval I, then the partial sums f_K of the Fourier series for f converges pointwise for every $x \in I$ with

$$\lim_{K \to \infty} f_K(x) = \begin{cases} f(x) & \text{if } x \in I \setminus \{x_j\}_{j=1}^{\infty} \\ \frac{1}{2}\Big[\underbrace{\lim_{\xi \nearrow x_j} f(\xi)}_{f(x_j^-)} + \underbrace{\lim_{\xi \searrow x_j} f(\xi)}_{f(x_j^+)} \Big], & (x = x_j) \end{cases}$$



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(i) This applies in the following cases:

(a) f is odd and periodic with period $2L$,

$$I = [-L, L]$$

$$f_K = \sum_{j=1}^{K} b_j \sin\left(\frac{j\pi}{L} x\right)$$

$$b_j = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{j\pi}{L} x\right) dx$$

$$= \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{j\pi}{L} x\right) dx$$

(b) f is even and periodic with period $2L$

$$I = [-L, L]$$

$$f_K = a_0 + \sum_{j=1}^{K} a_j \cos\left(\frac{j\pi}{L} x\right)$$

a_j = (the usual).

[redacted]

(c) f is $2L$ periodic with

$$f_K(x) = \underbrace{a_0 + \sum_{j=1}^{K} a_j \cos\left(\frac{j\pi}{L}x\right)}_{\text{even part}} + \underbrace{\sum_{j=1}^{K} b_j \sin\left(\frac{j\pi}{L}x\right)}_{\text{odd part.}}$$

(ii) Piecewise smooth:

Each $x \in I \setminus \{x_j\}_{j=1}^{\infty}$ determines an interval (x_j, x_k)

↑ singular points

with $x \in (x_j, x_k)$ and $f \in \boxed{C^0(x_j, x_k)}$ ← The set of continuous real valued functions on (x_j, x_k)

technically $f|_{(x_j,x_k)} \in C^0(x_j, x_k)$

↑ restriction of f

$f|_{(x_j,x_j)}$ extends with $f(x_j^+) = \lim_{\xi \searrow x_j} f(\xi)$ and [redacted]

- still saying what "piecewise smooth" means...

$f\big|_{(x_j,x_k)}$ extends with $f(x_j^+) = \lim\limits_{\xi \searrow x_j} f(\xi)$ and $f(x_k^-) = \lim\limits_{\xi \nearrow x_j} f(\xi)$

AND the extension $\bar{f}\big|_{(x_j,x_k)}$ satisfies

$$\bar{f}\big|_{(x_j,x_k)} \in \boxed{C^0[x_j,x_k]}$$

$\in I \setminus \{x_j\}_{j=1}^{\infty}$

AND

$f'(x)$ exists for each $x \in (x_j,x_k)$

with $f'\big|_{(x_j,x_k)} \in C^0(x_j,x_k)$ and f' [redacted]

extends: $\bar{f}'(x_j) = \lim\limits_{\xi \searrow x_j} f(\xi)$, $\bar{f}'(x_k) = \lim\limits_{\xi \nearrow x_k} f(\xi)$ [redacted]

$\bar{f}' \in C^0[x_j,x_k]$. $\xi \nearrow x_k$

<u>Notation</u>: The condition $f' \in C^0(a,b)$ is denoted by $f \in \boxed{C^1(a,b)}$

The collection of all "continuously differentiable" functions on the interval (a,b).

... $C^k(a,b)$

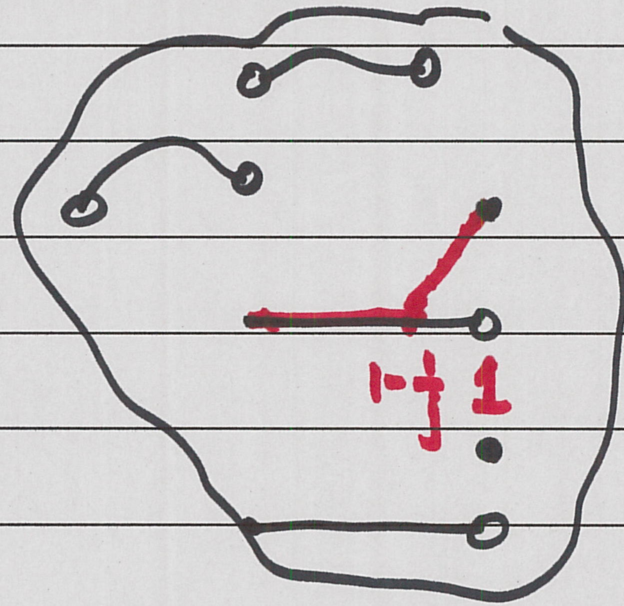
[redacted]

Other (semi-interesting) things in Chapter 3 of Haberman

<u>1</u>. If $f: I \to \mathbb{R}$ is <u>piecewise smooth</u>.

<u>and</u> $f \in C^0(I)$.

"continuous function with corners"



<u>Then</u>
(a) f_k' is the partial sum for the Fourier series of f'
(always true) <u>and</u>

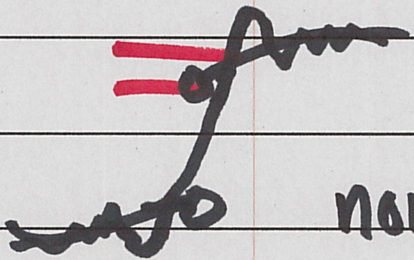
(b) $$\lim_{k \to \infty} f_k'(x) = \frac{1}{2}\left[\lim_{\xi \nearrow x} f'(\xi) + \lim_{\xi \searrow x} f'(\xi)\right]$$

[redacted]

2. Gibb's phenomenon

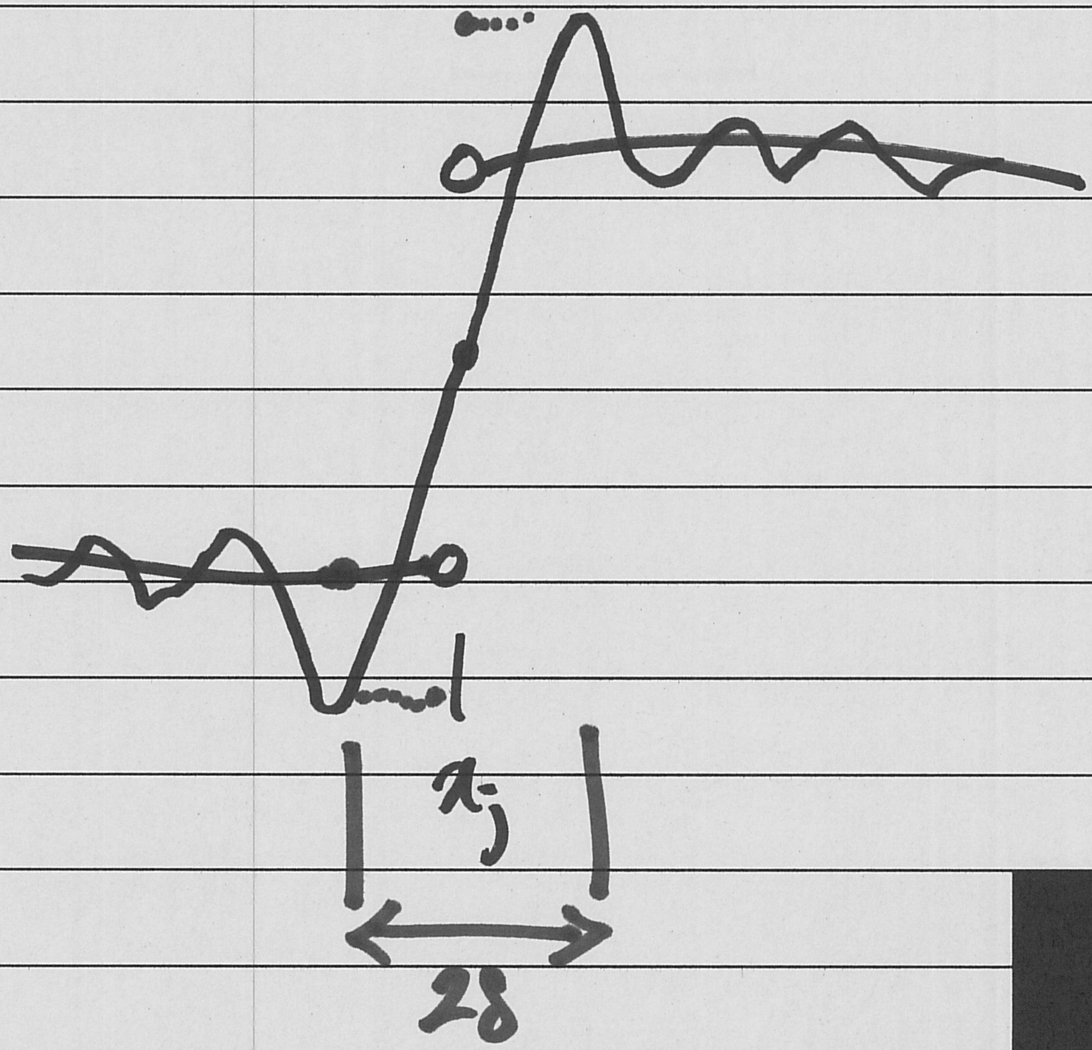
contrast: If f is piecewise smooth and $f \in C^0(I)$, then $f_K \to f$ pointwise uniformly.

Gibb's: Fourier series never converge uniformly near a discontinuity.



non-uniform overshoot.

[redacted]



[redacted]

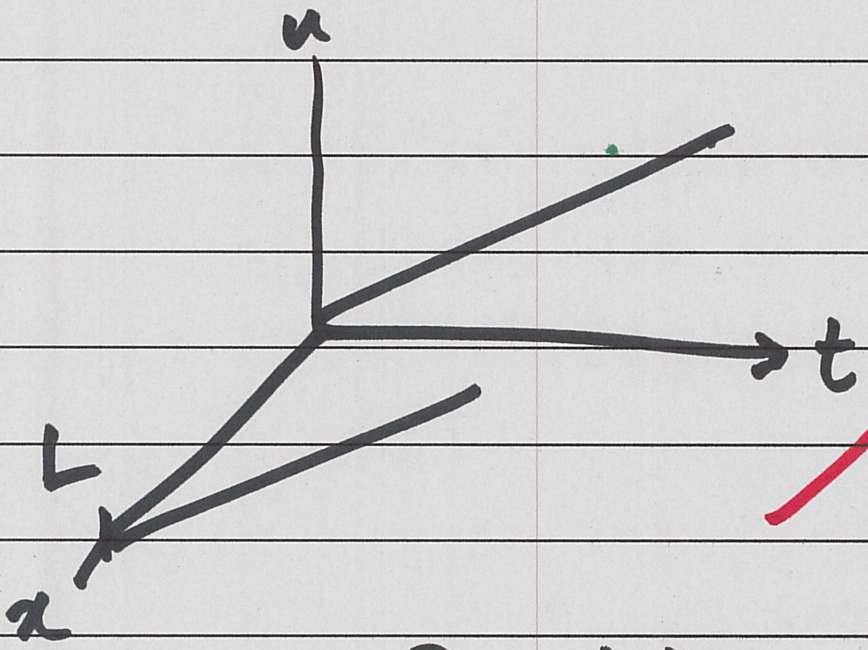
o Weak Maximum Principle

o $\begin{cases} u_t = u_{xx} \text{ on } (0,L)\times(0,T) \\ u(x,0) = 0 \\ u(0,t) = t = u(L,t) \end{cases}$

What we Know:

$u(x,t) = t$

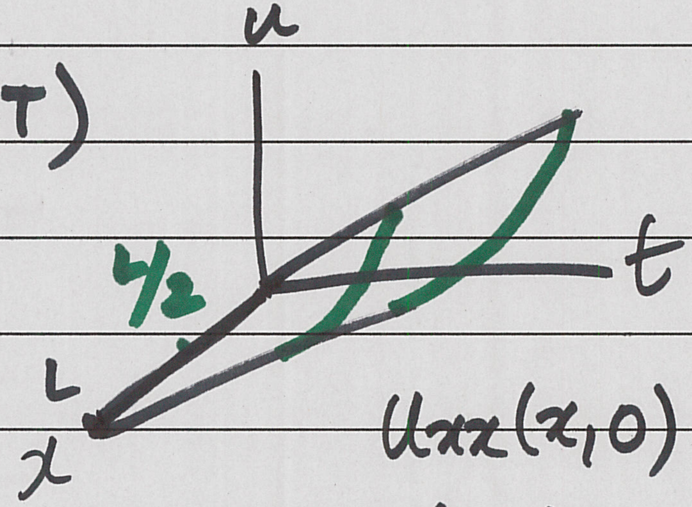
does not work.



o Wave Equation

[redacted]

$$\begin{cases} u_t = u_{xx} \quad \text{on } (0,L)\times(0,T) \\ u(x,0) = 0 \\ u(0,t) = t = u(L,t) \end{cases}$$

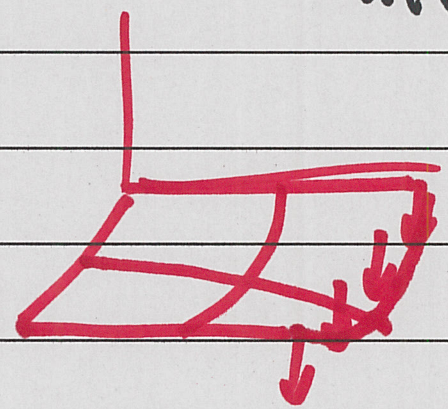


$u_{xx}(x,0) = 0$

$u_t(0,t) = 1 = u_t(L,t)$

If $w = u - t$, then

$$\begin{cases} w_t = w_{xx} - 1 \\ w(x,0) = 0 \\ w(0,t) = 0 = w(L,t) \end{cases}$$



What we might expect:

$0 < u(x,t) < t$ for $t > 0$.
$0 < x < L$

$w_t < 0$ for $t > 0$, $0 < x < L$.

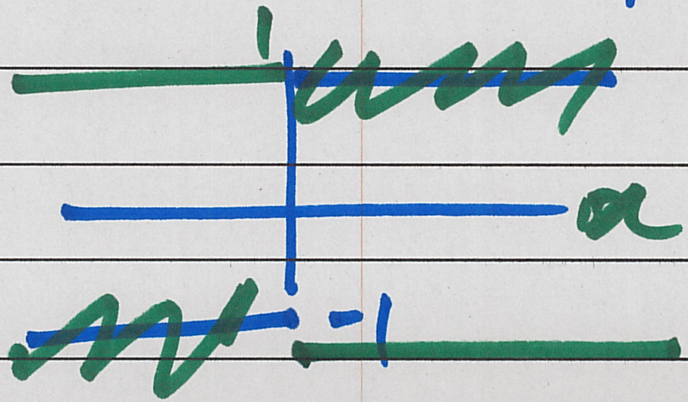
[redacted]

We like $A_j(x) = \sin \frac{j\pi}{L} x$

$$\begin{cases} W_t = W_{xx} - 1 \\ W(x,0) = 0 \\ \boxed{W(0,t) = 0 = W(L,t)} \end{cases}$$ on $(0,L) \times (0,T)$

$B(t) \sin \frac{j\pi}{L} x \leftarrow \boxed{A_j(0) = 0 = A_j(L)}$

Expand the constant function $f(x) = -1$.



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