MATH 4581 Lecture 15 Thursday October 14, 2021

LAST TIME : MATHY WORDS "OPEN" "CLOSED"

...

More: "interior", "connected".

Euclidean Point set topology.

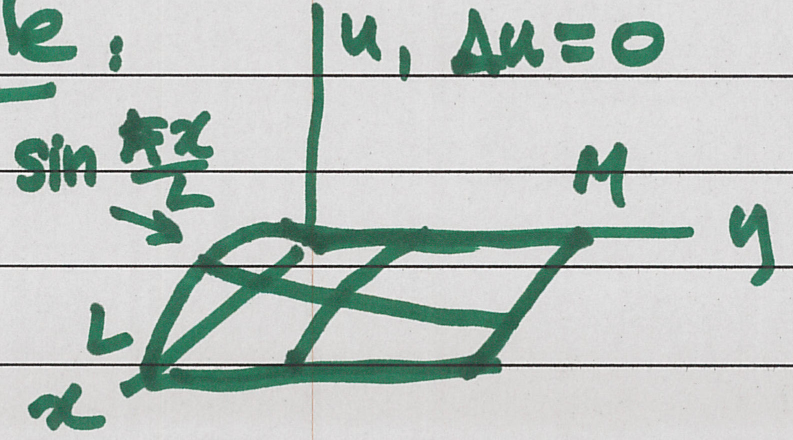
Assignments 4 and 5: through Chapter 3 Fourier Series (Haberman)

Separated variables: FORM $u = A(x)\,B(y)$

or $= A(x)\,B(t)$

or $= A(x)\,B(y)\,C(t)$

Example:



[redacted]

× Chapter 3 : Fourier Series

× More separated variables / superposition

People / Dates

Fourier (1768 - 1830)

$$f(x) \overset{?}{=} \underset{\text{Re}}{\uparrow} \sum_{j=1}^{\infty} a_j \, e^{\overset{\sqrt{-1}}{\downarrow}\, ijx}$$

e^{ijx} ↑ cosines / sines.

Dirichlet (1805 - 1859)
Dini (1845 - 1918)

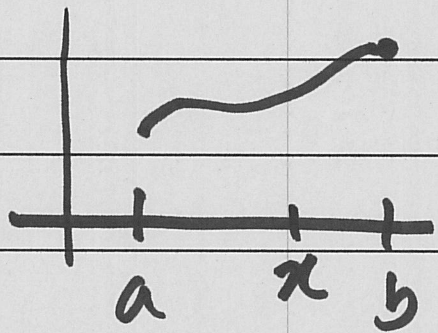
Lebesgue (1875 - 1941)

Hilbert (1862 - 1943)

Riesz (1880 - 1956) ←
Fischer (1875 - 1954)

[redacted]

Pointwise Convergence:



$f_K : (a,b) \longrightarrow \mathbb{R}$

a sequence of functions

$$\underbrace{\sum_{j=1}^{K} a_j \sin \frac{j\pi}{L} x}_{\substack{\text{Partial sum} \\ f_K(x)}} \xrightarrow{K \nearrow \infty} f(x) \quad (??)$$

[redacted]

Definition of convergence

Given a topology (on a set of functions) (which is a collection of "open" sets), f_K <u>converges</u> to f if given any open set U with $f \in U$, there is some N such that $f_K \in U$ for $K > N$.

If you pick the right topology, this is equivalent to

$\lim_{K \to \infty} f_K(x) = f(x)$ for each x.

(Pointwise convergence)

[redacted]

Many topologies are metric topologies

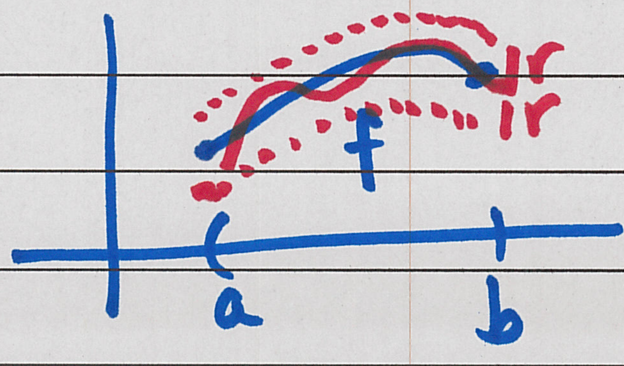
↑ open sets from open balls from a distance/metric

Urisohn Metrization Theorem
Nagata - Smirnov Metrization Thm

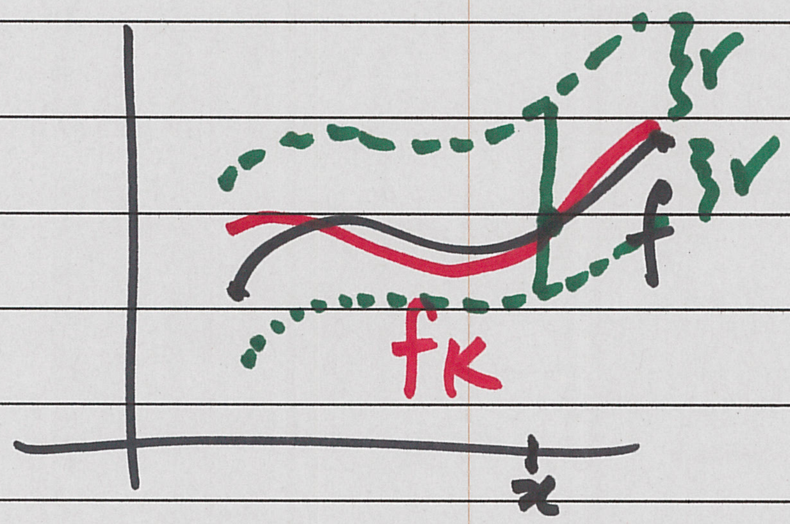
↑ Many

The topology of pointwise convergence is NON metrizable.

"like" the maximum.



$d(f,g) = \sup_{x \in (a,b)} |f(x)-g(x)|$ [redacted]



f_K converges to f in the metric top.

given any $r > 0$, there is some N such that $K > N$ implies $f_K \in B_r(f)$.

Does this imply pointwise convergence?

Yes!

[redacted]

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|$$

I ↑ interval

is called the <u>uniform metric</u> or the L^∞ metric

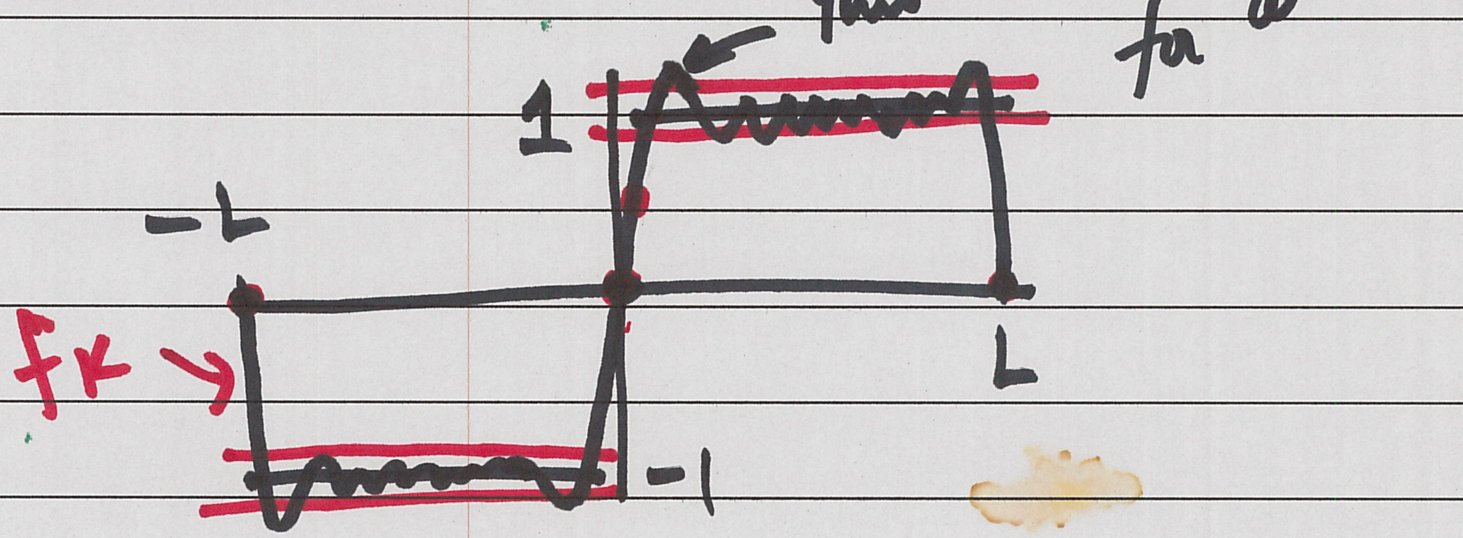
↑ Lebesgue

convergence with respect to this metric topology is called <u>uniform convergence</u>.

<u>Theorem</u> The uniform limit of continuous functions is continuous.

[redacted]

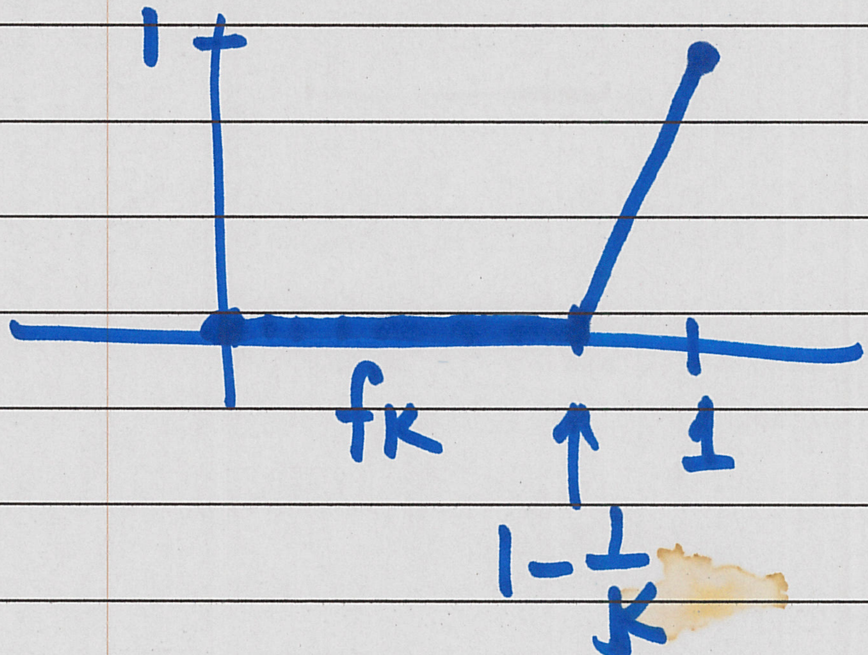
Gibb's Phenomenon:

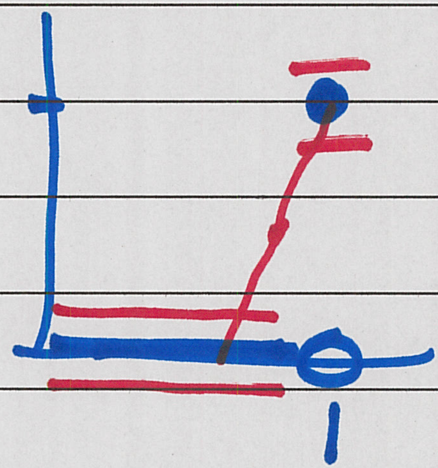


$$f(x) = \begin{cases} 0, & x = -L, 0, L \\ -1, & -L < x < 0 \\ 1, & 0 < x < L \end{cases}$$

[redacted]

Example





Pointwise Limit? $f(x) = \lim_{K \to \infty} f_K(x)$

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$$

[redacted]

Fourier's Theorem (ugly)

If f is "piecewise smooth" meaning there are countably many points $\{x_j\}_{j=1}^{\infty} \subseteq I$ for which any point $x \in I \setminus \{x_j\}_{j=1}^{\infty}$ lies in an open interval

$$x \in (x_j, x_k)$$

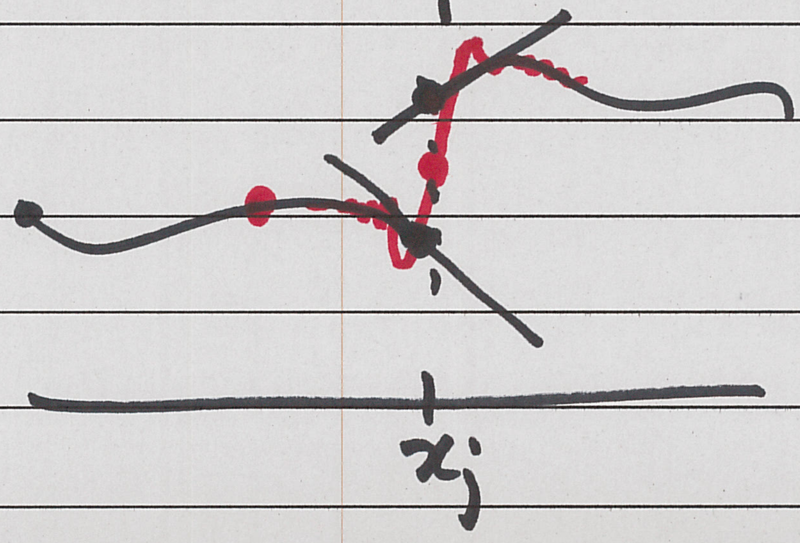
and f is continuous on (x_j, x_k)

and f' has a continuous extension to $[x_j, x_k]$,

then [redacted]

The Fourier series for f has a pointwise limit

$$\lim_{K\to\infty} f_K(x) = \begin{cases} f(x) & \text{for } x \in I \setminus \{x_j\}_{j=1}^{\infty} \\ \dfrac{f(x_j^-)+f(x_j^+)}{2}, & x \in \{x_j\}_{j=1}^{\infty} \end{cases}$$



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