MATH 4581 Thursday October 7, 2021

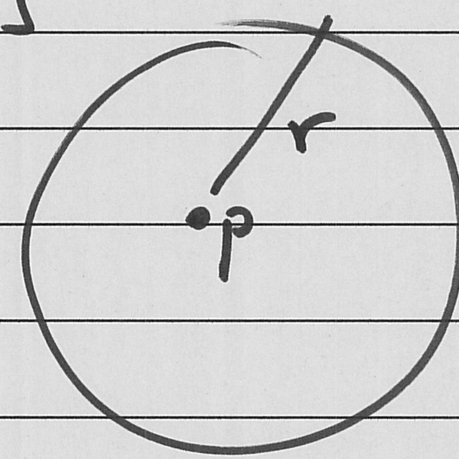
Lecture 14 open and closed sets (and many other things).

$$B_r(p) = \{ x : d(x,p) < r \}$$

$\uparrow$ distance

$\uparrow$ defined for $r > 0$

"open ball with center p and radius r"



<u>Example</u>: $L^2(0,1) = \{ f : \int_{(0,1)} |f|^2 < \infty \}$ [redacted]

$$d(f,g) = \sqrt{\int_0^1 |f-g|^2 \, dx}$$

$\uparrow$ a distance between two functio[redacted]

Abstract properties of a distance:

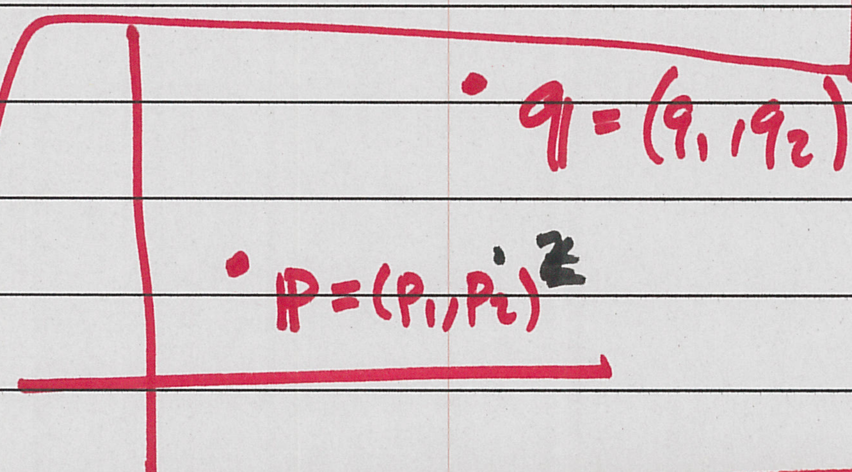
(1) $d(p,q) \geq 0$ with equality only if $p=q$. (positive definite)

(2) $d(p,q) = d(q,p)$ (symmetric)

(3) $d(p,q) \leq d(p,z) + d(z,q)$ (triangle inequality)

$d(p,q) = |q_1 - p_1| + |q_2 - p_2|$

$\tilde{d}(p,q) = \sqrt{(q_1-p_1)^2 + (q_2-p_2)^2}$



Metric [redacted]

A set X with a distance = Space [redacted]

$B_r(p)$ is open.

proof: Let $a \in B_r(p)$.

a new radius

Let $c = r - d(p,a)$. Note that $c > 0$.

We need to show $B_c(a) \subseteq B_r(p)$. ✓

Let $x \in B_c(a)$. Then $d(x,a) < c$.

$$
\begin{aligned}
\text{So } d(x,p) &\leq d(x,a) + d(a,p) \\
&< c + d(a,p) \\
&= r - d(p,a) + d(a,p) \\
&= r. \quad \text{So } x \in B_r(p).
\end{aligned}
$$

[redacted]

<u>Exercise</u> Show any union of open sets is open.

<u>Exercise</u> The intersection of any two open sets is open.

<u>Exercise</u> Show that not every intersection of open sets is open.

An intersection of open sets which is <u>not</u> open.

In $\mathbb{R}^1$, $\bigcap_{j=1}^{\infty} \left(0, 1+\frac{1}{j}\right) = (0,1]$

[redacted]

<u>Definition</u>: A set is closed if its complement is open.

<u>Example</u> $[0,1] \subseteq \mathbb{R}^1$ is closed.

$$[0,1]^c = \mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$$

open.

Q: Is $\mathbb{R}$ open? Yes!

Q: Is $\mathbb{R}$ closed?

empty set

$$\mathbb{R}^c = \mathbb{R} \setminus \mathbb{R} = \phi$$

Is the empty set open?

Yes.

[redacted]

Is $[0,1)$ open? No!

Is $[0,1)$ closed? No. Why?

$$[0,1)^c = (-\infty,0) \cup [1,\infty)$$

BOUNDARY:

EXERCISE: Any intersection of closed sets is closed.

$$\left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c \quad \text{(deMorgan's Law)}$$

Consequence: Given any set A, there is a (unique) smallest closed set F with $F \supseteq A$.

[redacted]

Given any set A There is a smallest closed set containing A.

This set is called The <u>closure of A</u> and is denoted

$$\overline{A}$$

Exercise: Show a set A is <u>closed</u> if and only if $\overline{A} = A$.

$$\overline{B_r(p)} = \{x : d(x,p) \leq r\}$$

$$\partial A = \overline{A} \cap \overline{A^c}$$

↑ The <u>boundary</u> of A

[redacted]

Weak Maximum Principle:

$$\max_{x \in \overline{\mathcal{U}}} u(x) = \max_{x \in \partial \mathcal{U}} u(x)$$

↑

Extreme Value Theorem: A continuous function on a closed and bounded subset of $\mathbb{R}^n$ always achieves its maximum value.

$\partial \mathcal{U}$ is closed. Why?

$\partial \mathcal{U} = \overline{\mathcal{U}} \cap \overline{\mathcal{U}^c}$

an intersection of closed sets.

[redacted]

Strong Maximum Principle:

$$u(x_0) < \max_{x \in \partial U} u \quad \text{for } x_0 \in U$$

unless $u \equiv \text{const}.$

Maybe it should be easy to prove the weak maximum principle:

$\underline{n=2}$ $\quad \Delta u = 0.$

$u(p) > M$

$u(x_*) = \max_{\partial U} u = M$

[redacted]

$\underline{\text{Consider}}$ $\quad w(x,y) = -(x-p_1)^2 - (y-p_2)^2$