Lecture 11 Tuesday Sept. 28, 2021

MATH #581

- Calculus of Variations ("finish")
- Assignment 2 Problem on transport
  3.
- Problem 5. (chain rule).

[redacted]

$\mathcal{F} : A \to \mathbb{R}$

Remember a point $x_0 \in (a,b)$ is a
<u>stationary point</u> for $f : (a,b) \to \mathbb{R}$
if $f'(x_0) = 0$.

A function $u \in A$ is <u>stationary</u> for $\mathcal{F}$
if $\delta \mathcal{F}_u[\phi] = 0$ for "all" $\phi \in \mathcal{V}$.

[redacted]

What is the Lagrangian for Dirichlet Energy?

$$\mathcal{D}[u] = \int_{\mathcal{U}} |Du|^2$$

Answer!

$$F(\mathbb{P}) = |\mathbb{P}|^2.$$

$$\delta\mathcal{D}_u[\phi] = \frac{d}{d\varepsilon} \int_{\mathcal{U}} \underbrace{|Du + \varepsilon D\phi|^2}_{= |D(u+\varepsilon\phi)|^2} \Big|_{\varepsilon=0}$$

derivatives w.r.t. x

[redacted]

$$\frac{d}{d\varepsilon}\int_U |Du+\varepsilon D\phi|^2 \Big|_{\varepsilon=0}$$

ε dependence

$$\sum_{j=1}^{n}\left(\underbrace{\cancel{D}u_{x_j}+\varepsilon\phi_{x_j}}\right)^2$$

$$= \int_U \frac{\partial}{\partial\varepsilon} |Du+\varepsilon D\phi|^2 \Big|_{\varepsilon=0}$$

$$= \int_U \sum_{j=1}^{n} 2\left(u_{x_j}+\varepsilon\phi_{x_j}\right)\phi_{x_j} \Big|_{\varepsilon=0}$$

$$= 2\int_U Du\cdot D\phi$$

[redacted]

$$\delta \mathcal{D}_u[\phi] = 0 \quad \text{for all } \phi$$

$$\iff \int_U \underline{Du \cdot D\phi} = 0 \quad \text{for all } \phi.$$

(Integration by parts = product rule for divergence of a scaled field + divergence thm.)

$$\operatorname{div}(\phi Du) = \phi \underbrace{\operatorname{div} Du}_{= \Delta u} + D\phi \cdot Du$$

$$\iff \int_U \operatorname{div}(\phi Du) - \int_U \Delta u \, \phi = 0$$

for all ϕ.

[redacted]

$$\Longleftrightarrow \underbrace{\int_{\mathcal{U}} \operatorname{div}(\phi\, Du)}_{\displaystyle \int_{\partial \mathcal{U}} \phi\, Du \cdot \mathbf{n} \;=\; 0} - \int_{\mathcal{U}} \Delta u\, \phi = 0 \quad \text{for all } \phi.$$

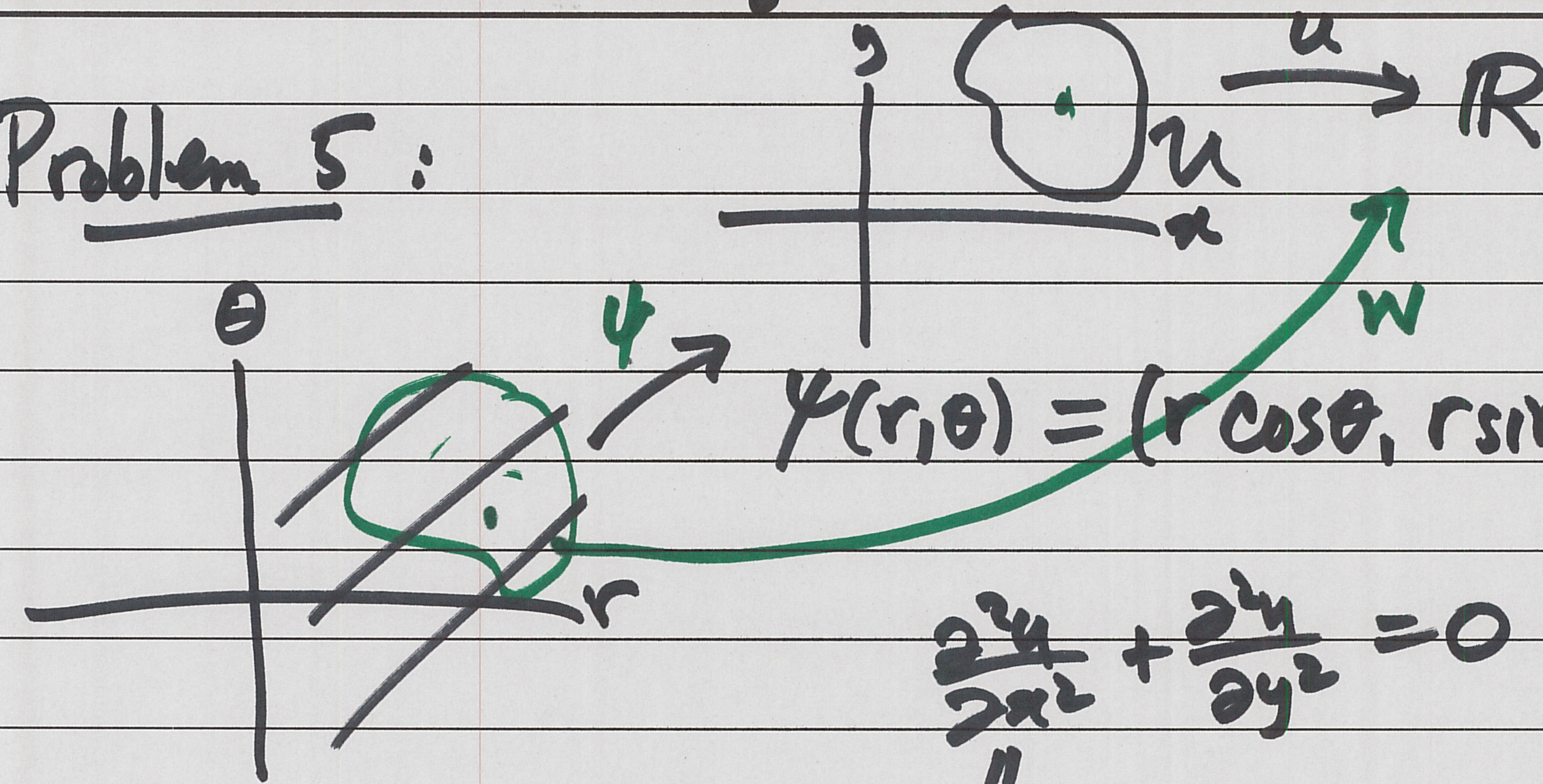
because $\phi|_{\partial \mathcal{U}} \equiv 0$.

$$\Longleftrightarrow \int_{\mathcal{U}} \Delta u\, \phi = 0 \text{ for all } \phi$$

Fundamental Lemma of COV

$$\Longleftrightarrow \Delta u = 0 \text{ on } \mathcal{U}.$$

Problem 5 :



$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Q: If u satisfies $\Delta u = 0$, then

what PDE does $w(r,\theta)$

$= u \circ \psi(r,\theta)$ satisfy.

("Laplace's equation in polar coords.")

[redacted]

If $u : \mathcal{U} \times [0,T) \to \mathbb{R}$
$\uparrow$ heat domain

with $\mathcal{U} \in \mathbb{R}^2$, then $u = u(\underset{\underset{(x,y)\in\mathcal{U}}{\shortparallel}}{x}, t)$

Then we can take
$$\begin{aligned} w &= w(r,\theta,t) \\ &= u(\psi(r,\theta), t) \\ &= u(r\cos\theta, r\sin\theta, t) \end{aligned}$$

and ask if $u_t = \Delta u$ what PDE does w satisfy?

[redacted]

$W_t = U_t$

$$W(r,\theta,t) = u(r\cos\theta, r\sin\theta, t)$$

$$W_r = U_x \cdot \cos\theta + U_y \cdot \sin\theta$$

1-D

$$(g\circ f)' = g'\circ f \cdot f'$$

$$W_\theta = U_x(-r\sin\theta) + U_y \cdot (r\cos\theta)$$

<u>Note:</u> $\begin{pmatrix} W_r \\ W_\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}} \begin{pmatrix} U_x \\ U_y \end{pmatrix}$ [redacted]

$\Rightarrow \begin{pmatrix} U_x \\ U_y \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} W_r \\ W_\theta \end{pmatrix}$ [redacted]

$u_x(r\cos\theta, r\sin\theta)$

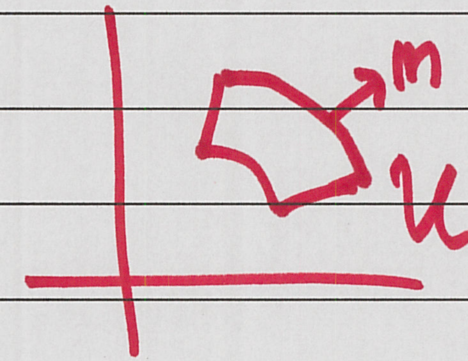
$$W_\theta = \underbrace{-r\sin\theta}\ \underbrace{u_x} + \underbrace{r\cos\theta}\ \underbrace{u_y}$$

$$\begin{aligned} W_{\theta\theta} = & -r\cos\theta\ u_x \\ & -r\sin\theta\left[-r\sin\theta\ u_{xx} + u_{xy}\ r\cos\theta\right] \\ & + (-r\sin\theta)\,u_y \\ & + r\cos\theta\left[-r\sin\theta\ u_{xy} + r\cos\theta\ u_{yy}\right] \end{aligned}$$

$$\begin{aligned} = & -r\cos\theta\ u_x - r\sin\theta\ u_y \\ & + r^2\sin^2\theta\ u_{xx} \\ & - 2r^2\cos\theta\sin\theta\ u_{xy} \\ & + r^2\cos^2\theta\ u_{yy} \end{aligned}$$

[redacted]

$$W_{\theta\theta} = -r\cos\theta\, u_x - r\sin\theta\, u_y$$
$$+ r^2 \sin^2\theta\, u_{xx}$$
$$- 2r^2 \cos\theta \sin\theta\, u_{xy}$$
$$+ r^2 \cos^2\theta\, u_{yy}$$



$W_r = \cos\theta\, u_x + \sin\theta\, u_y$

$$W_{rr} = \cos^2\theta\, u_{xx} + 2\cos\theta\sin\theta\, u_{xy} + \sin^2\theta\, u_{yy}$$

To <u>find</u>: $u_{xx} + u_{yy}$

$$W_{rr} + \frac{1}{r^2} W_{\theta\theta} = u_{xx} + u_{yy}$$
$$+ \text{(other stuff)}$$

[redacted]