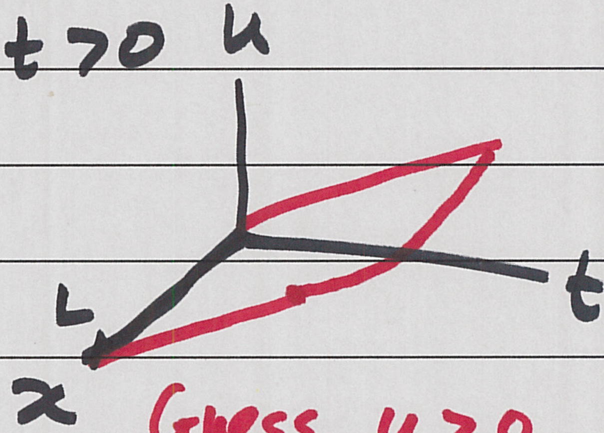
MATH 4581 Lecture 8 Thursday Sept. 16, 2021

Last time: Assignment 1, Problem 6

$$\longrightarrow \begin{cases} u_t = u_{xx} \quad \text{on } (0,L)\times(0,T) \\ u(0,t) = t = u(L,t), \quad t>0 \\ u(x,0) = 0, \quad x \in (0,L) \end{cases}$$

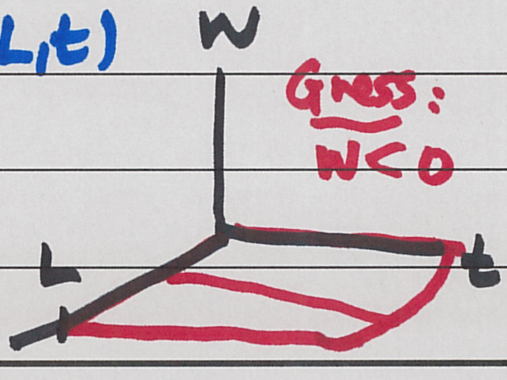


Guess $u > 0$
$u < t$

equivalent:

forced.

$$\begin{cases} w_t = w_{xx} - 1 \quad \text{on } (0,L)\times(0,T) \\ w(0,t) = 0 = w(L,t) \\ w(x,0) = 0 \end{cases}$$



Guess: $w<0$

$$\boxed{w = u - t}$$

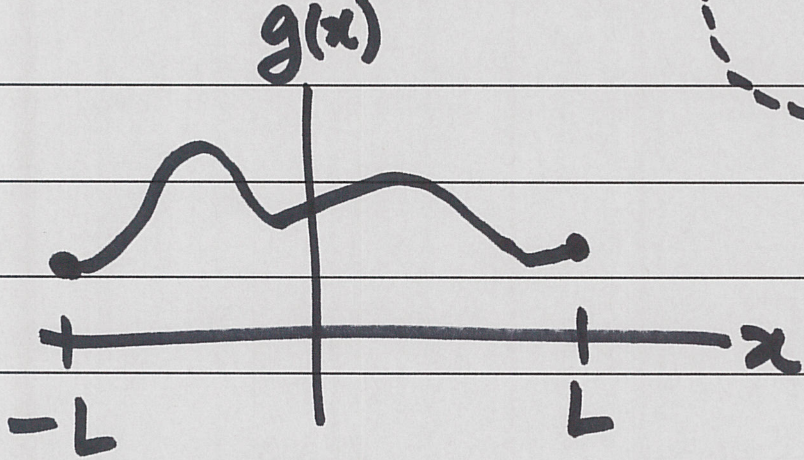
[redacted]

Think: These are interesting and reasonable problems (with nice physical interpretations) and I don't know how to solve them.

Grad Students: Go for it!

[redacted]

§ 2.4.2 heat conduction in a ring.



initial (periodic) temp. distribution.

PDE $u_t = u_{xx}$ (or $u_t = k u_{xx}$)

Periodic boundary conditions :

$$\begin{cases} u(-L,t) = u(L,t) \\ u_x(-L,t) = u_x(L,t) \end{cases}$$

[redacted]

Separation of Variables :

$$u = A(x)\,B(t)$$

$$AB' = k\,A''B$$

$$\frac{A''}{A} = \frac{B'}{kB} = \lambda$$

Boundary Conditions :

$$A(-L)\,B(t) = A(L)\,B(t)$$

$$\begin{cases} A(-L) = A(L) \\ A'(-L) = A'(L) \end{cases}$$

$$\begin{cases} A'' = \lambda A \\ A(-L) = A(L) \\ A'(-L) = A'(L) \end{cases}$$

Strum-Liouville Problem .

[redacted]

$$\begin{cases} A'' = \lambda A \\ A(-L) = A(L) \\ A'(-L) = A'(L) \end{cases}$$

<u>CASE 1</u> $\lambda = -\mu^2 < 0.$

$$A = a\cos\mu x + b\sin\mu x$$

$$A(-L) = a\cos\mu L - b\sin\mu L$$
$$A(L) = a\cos\mu L + b\sin\mu L$$

$$A(L) = A(-L) \implies b\sin\mu L = 0.$$

CASE 1a

$b = 0$

<u>or</u>

Case 1b

$\sin\mu L = 0$

If $b=0$, $A = a\cos\mu x$, $A' = -a\mu\sin\mu x$

$A'(-L) = A'(L) \implies \sin\mu L = 0.$

[redacted]

$$\mu_j = \frac{j\pi}{L} \quad , \; j = 1, 2, 3, \ldots$$

$$A_j = a_j \cos \frac{j\pi}{L} x \quad , \quad \tilde{A}_j = b_j \sin \frac{j\pi}{L} x$$

$$B_j = e^{-\frac{k j^2 \pi^2}{L^2} t}$$

$$u_j = a_j e^{-\frac{k j^2 \pi^2}{L^2} t} \cos \frac{j\pi}{L} x \; , \; \tilde{u}_j = b_j e^{-\frac{k j^2 \pi^2}{L^2} t} \sin \frac{j\pi}{L} x$$

<u>CASE 2</u> $\lambda = 0$, $A = ax + b$

$u_0 = \text{const.}$

[redacted]

<u>CASE 3</u> $\lambda = \mu^2 > 0$

$A = a \cosh \mu x + b \sinh \mu x.$

$\longrightarrow$ no solutions.

We have separated variables "solutions"

$u_0 = \text{const.}$

$u_j = a_j \, e^{-k \frac{j^2 \pi^2}{L^2} t} \cos \frac{j\pi}{L} x$

$\tilde{u}_j = b_j \, e^{-k \frac{j^2 \pi^2}{L^2} t} \sin \frac{j\pi}{L} x$

satisfy $u_t = k u_{xx}$, & boundary cond. (periodic)

but maybe not $u(x,0) = u_0(x) = g(x)$

Try a superposition :

[redacted]

$$u = u_0 + \sum_{j=1}^{\infty}\left(a_j \cos\frac{j\pi}{L}x + b_j \sin\frac{j\pi}{L}x\right)e^{-k\frac{j^2\pi^2}{L^2}t}$$

$$u(x,0) = u_0 + \sum_{j=1}^{\infty}\left(a_j \cos\frac{j\pi}{L}x + b_j \sin\frac{j\pi}{L}x\right) \overset{?}{=} g(x)$$

Think:

Basis : $\{ 1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \dots$
$\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \dots \}$

hope: orthogonal. ✓

Multiply by a basis ~~element~~ vector and integrate [redacted]

$$g(x) = a_0 + \sum_{j=1}^{\infty}\left(a_j \cos\frac{j\pi}{L}x + b_j \sin\frac{j\pi}{L}x\right)$$

$$\int_{-L}^{L} g(x)\,dx = 2L a_0 \Rightarrow a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,dx$$

$$\rightarrow \int_{-L}^{L} g(x)\cos\frac{j\pi}{L}x\,dx = a_j \underbrace{\int_{-L}^{L}\cos^2\frac{j\pi}{L}x\,dx}_{L}$$

$$a_j = \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{j\pi}{L}x\,dx$$

$$b_j = \frac{1}{L}\int_{-L}^{L} g(x)\sin\frac{j\pi}{L}x\,dx$$

[redacted]

Laplace's Equation:

- Equilibrium solutions u^* of $u_t = \Delta u$

  satisfy $\boxed{\Delta u^* = 0}$ ← u^*

  all spatial.

- GUESS: $\lim_{t \to \infty} u(x,t) = u^*(x)$

- Separation of variables can work when $n > 1$ sometimes.

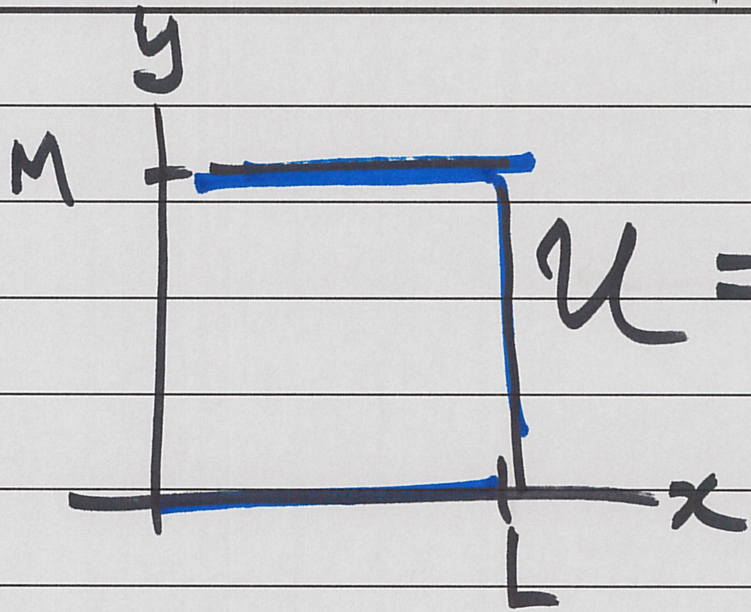
[redacted]

- Nice derivation using <u>Calculus of variations</u>

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (n=2)$$

$u = u(x,y)$

[redacted]



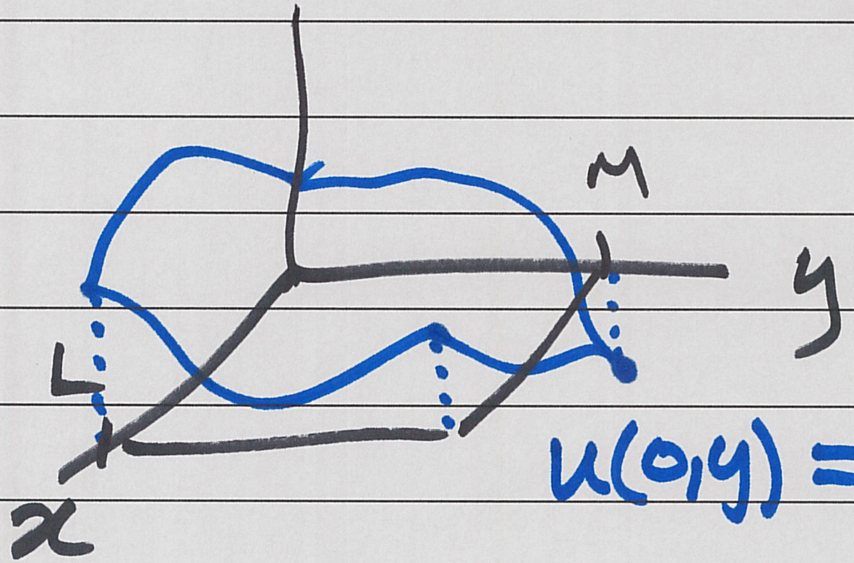
$\mathcal{U} = (0,L) \times (0,M)$

$$\Delta u = 0$$

Boundary conditions

$u(x,0) = g_0(x), \quad 0<x<L$

$u(x,M) = g_1(x), \quad 0<x<L$

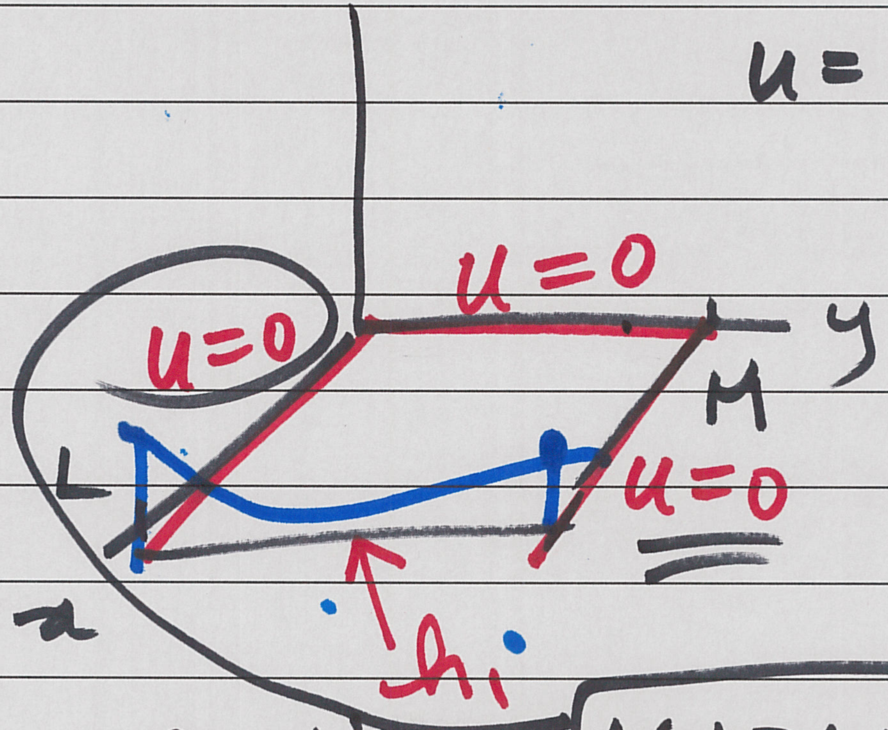


$u(0,y) = h_0(y)$
$0<y<M$

$u(L,y) = h_1(y)$
$0<y<M.$

[redacted]

Solve instead with 3 out of 4 homogeneous boundary conditions:



$$u = A(x)\,B(y)$$

$$A''B + B''A = 0$$

$$\frac{A''}{A} = -\frac{B''}{B} = \lambda$$

Boundary Conditions: $\boxed{A(x)\,B(0) = 0}$

$A(x)\,B(M) = 0$

[redacted]