MATH 4581 Tuesday September 7, 2021

Lecture 5

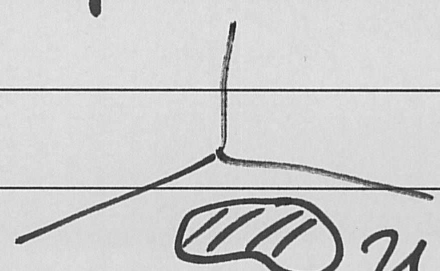
Last time: $\frac{\partial}{\partial t}(c\rho u) = \operatorname{div}(K\,Du) + Q$

$$\left[\; u_t = k\,\Delta u + f \quad \text{or} \quad u_t = \Delta u \;\right]$$

divergence Theorem:

spatial domain $\mathcal{U} \subseteq \mathbb{R}^n$

$$\int_{\mathcal{U}} \operatorname{div} \mathbb{V} = \int_{\partial \mathcal{U}} \mathbb{V} \cdot \vec{n}$$



$\mathcal{U} \subseteq \mathbb{R}^2$

[redacted]

$\boxed{\text{TO DO/TALK ABOUT}}$

- Regularizing properties of elliptic & parabolic PDE (infinite propagation speed)

- <u>Transport Equations</u>

<u>motion of substrate/medium</u>

<u>Today</u>: Separation of Variables.

Not separable ODE

$$f(y)\,y' = g(x)$$

[redacted]

Chapter 2 Method of Separation of Variables.

2.4.1 Insulated ends

$$\begin{cases} u_t = k\, u_{xx} & \text{on } (0,L)\times(0,T) \\ u_x(0,t) = 0 = u_x(L,t), & t > 0 \\ u(x,0) = g(x), & 0 < x < L \end{cases}$$

$u(x,t) = A(x)\,B(t)$ ← just try this.

PDE $AB' = k\, A'' B$

$B' = \dfrac{dB}{dt}$, $A'' = \dfrac{d^2A}{dx^2}$

[redacted]

$$A B' = k A'' B$$

$$\underline{\frac{B'}{B} = k \frac{A''}{A}} \quad \text{or} \quad \frac{1}{k}\frac{B'}{B} = \frac{A''}{A}$$

only a function of t

only a function of x.

$$\frac{B'}{B} = k \frac{A''}{A} = \lambda \text{ (const)}$$

separation constant

[redacted]

$B' = \lambda B$ and $A'' = \frac{\lambda}{k} A$

These look like ODE's — but they are not quite ODE's because λ is an unknown.

Sturm-Liouville Problem

To find possible solutions A, B, λ we need boundary values (initial values)

$A'(0)B(t) = 0 = A'(L)B(t)$

$$\begin{cases} A'(0) = 0 = A'(L) \\ A'' = (\lambda/k)A \end{cases} \quad \begin{cases} \text{TWO POINT} \\ \text{Boundary} \\ \text{conditions} \end{cases}$$

[redacted]

$$\begin{cases} A'' = (\lambda/k)A & \text{on } (0,L) \\ A'(0) = 0 = A'(L) \end{cases}$$

Initial condition from PDE $A(x)\underline{\underline{B(0)}} = g(x)$ ← It's not clean what this means for us. (set it aside.)

CASE 1 $\lambda/k < 0$
$= -\mu^2, \quad \mu > 0$

$\mu = \sqrt{-\frac{\lambda}{k}}$

$$\boxed{A'' = -\mu^2 A}$$

↳ $A(x) = a\cos\mu x + b\sin\mu x$
(general solution)

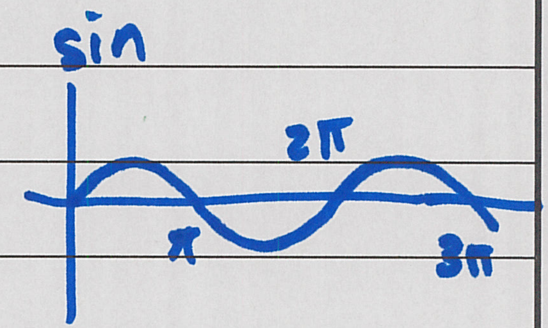
[redacted]

$$A(x) = a\cos\mu x + b\sin\mu x$$

$$A' = -a\mu\sin\mu x + \underline{\underline{b}}\,\mu\cos\mu x$$

$$A'(0) = 0 \Longrightarrow b\mu = 0 \Rightarrow b = 0.$$

$$A'(L) = -a\mu\,\underbrace{\sin(\mu L)}_{} = 0.$$



$$\mu L = j\pi\ ,\quad j = 1, 2, 3, 4, \ldots$$

$$\mu = \frac{j\pi}{L}$$

[redacted]

$$A'' = -\mu^2 A \,, \qquad \frac{\lambda}{k} = -\mu^2$$

$$A(x) = \underbrace{a \cos \mu x}\ \cancel{+ b \sin \mu x}$$

$$\mu_j = \frac{j\pi}{L} \,, \quad \lambda_j = -k \frac{j^2\pi^2}{L^2} \,, \quad j = 1, 2, 3, \ldots$$

$$\boxed{A_j(x) = a_j \cos\left(\frac{j\pi}{L} x\right)}$$

$$B_j' = \lambda_j B_j \Rightarrow \underline{B_j' = -k \frac{j^2\pi^2}{L^2} B_j}$$

<u>general solution</u>: $B_j(t) = e^{-k \frac{j^2\pi^2}{L^2} t}$

[redacted]