MATH 4581 Lecture 4 Sept. 2, 2021

- The heat equation in higher spatial dimensions

(- Method of Separation of variables)

<u>Last time</u> $\boxed{u_t = u_{xx}}$ (or $u_t = \underset{\text{diffusivity.}}{k}\, u_{xx}$)

1-D heat eqn.

<u>Initial condition</u> : $u(x,0) = u_0(x)$

<u>boundary conditions</u> : fixed temp.

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases}$$

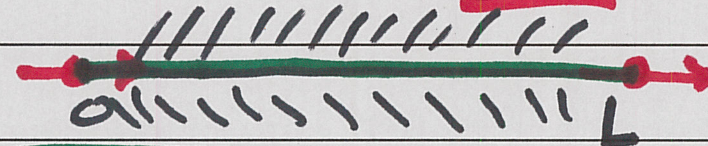
[redacted]

boundary conditions for <u>insulated ends</u>(?)

<u>1-D</u> <u>Fourier's Law</u> $\phi(x,t) = -K\,u_x(x,t)$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$$

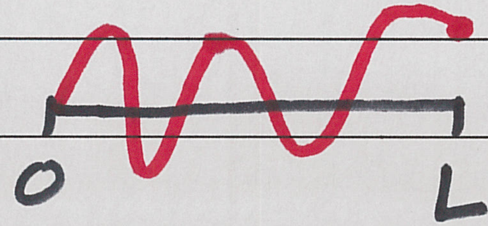
INSULATED <u>ENDS</u>



Take this and $\boxed{u_t = k\,u_{xx}}$

$u(x,0) = u_0(x)$

→ What happens?



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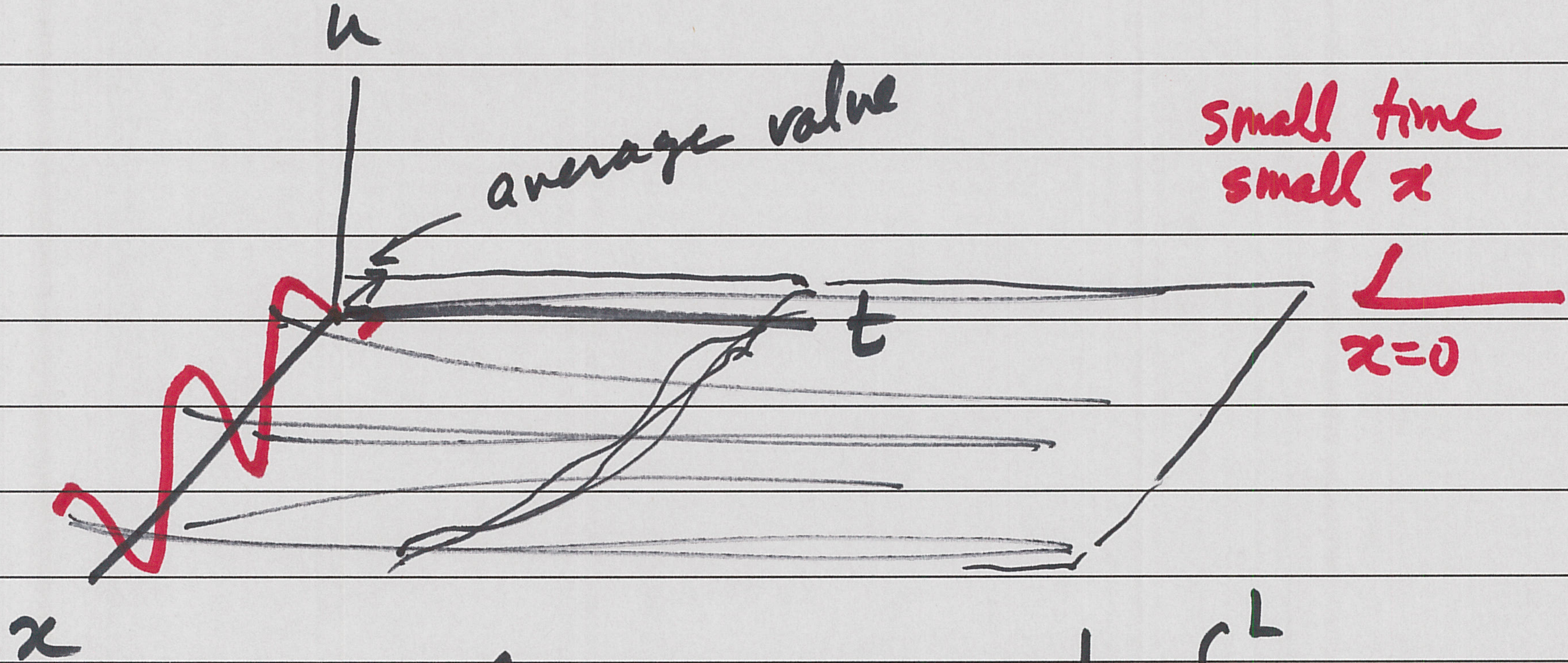


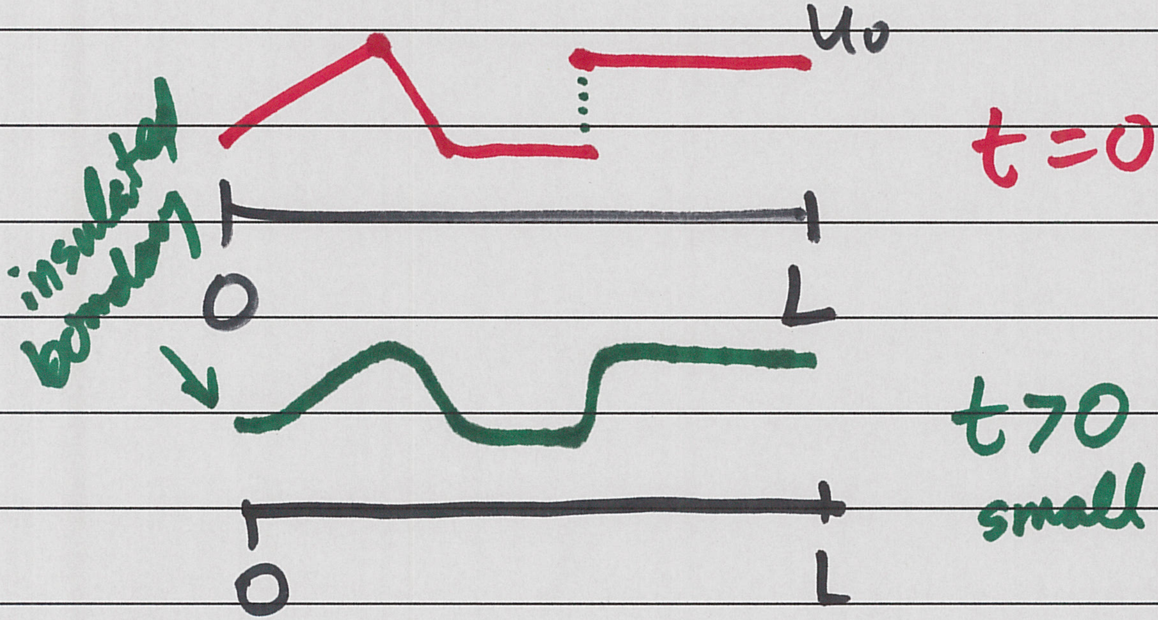
<u>Claim</u>: $\lim_{t \to \infty} u(x,t) = \frac{1}{L}\int_0^L u_0(x)\,dx$

$u_x(0,t) = 0, \quad t > 0$

[redacted]

Regularizing Parabolic PDE are instantaneously regularizing

Infinite proposation speed Parabolic equations can "move things" infinitely fast.

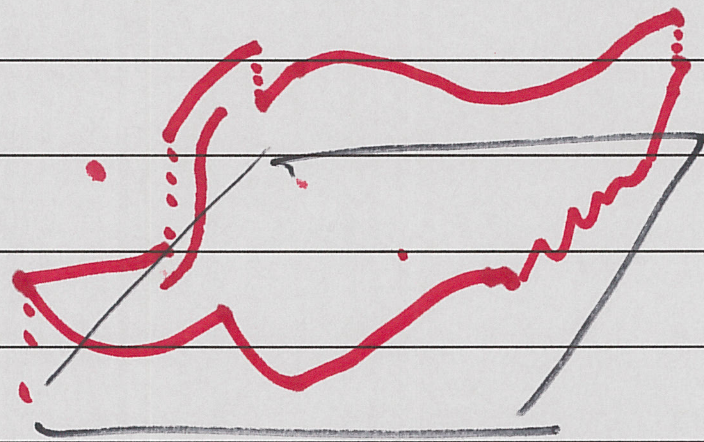


$$\begin{cases} u_t = u_{xx} \quad \text{for } (x,t) \in (0,L)\times(0,T) \\ u_x(0,t) = 0 = u_x(L,t),\ t>0 \\ u(x,0) = u_0 \end{cases}$$

[redacted]

$\Delta u = 0$ (elliptic)

<u>$n=2$</u> $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 .$



$u_t = \Delta u$

[redacted]

θ $\qquad$ $\mathcal{U} \subseteq \mathbb{R}^{n}$

$$[\theta] = \frac{[\text{energy}]}{L^3}$$

volumetric thermal energy density.

<u>New ideas</u> for $n > 1$

- Integration on sets. $\qquad \int_{\mathcal{U}} \theta = \int_{x \in \mathcal{U}} \theta$
- Fourier's Law

$$\vec{\Phi} = -K\,Du = -K\nabla u$$

spatial gradient

- Fundamental Theorem of Calculus

$\longrightarrow$ The Divergence Theorem.

[redacted]

Fundamental Theorem of Calculus

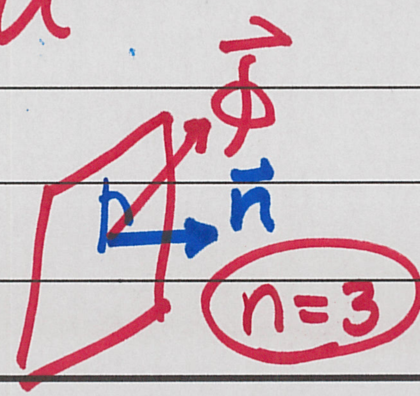
(a) $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

(b) $$\frac{d}{dx}\int_a^x g(\xi)\,d\xi = g(x)$$

$$\frac{d}{dt}\int_{\mathcal{U}} \theta = -\int_{\partial\mathcal{U}} \vec{\phi}\cdot\vec{n}$$

flux integral

boundary of $\mathcal{U}$



[redacted]

$$\frac{d}{dt}\int_{\mathcal{U}} \theta = - \int_{\partial \mathcal{U}} \vec{\phi} \cdot \vec{n} + \int_{\mathcal{U}} Q$$

$$\int_{\mathcal{U}} \theta_t = - \int_{\mathcal{U}} \operatorname{div} \vec{\phi} + \int_{\mathcal{U}} Q$$

flux field

The Divergence Theorem

$$\int_{\mathcal{U}} \operatorname{div} \mathbb{V} = \int_{\partial \mathcal{U}} \mathbb{V} \cdot \vec{n}$$

temporarily think $\operatorname{div} \mathbb{V} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$

[redacted]

$$\int_{\mathcal{U}} [\theta_t + \operatorname{div}\vec{\phi} - Q] = 0$$

$$\int_{R} [\theta_t + \operatorname{div}\vec{\phi} - Q] = 0 \quad \text{for all } R \subseteq \mathcal{U}.$$

$(R \subseteq \mathcal{U} \subseteq \mathbb{R}^n)$

$$\boxed{\theta_t + \operatorname{div}\vec{\phi} = Q}$$

[redacted]

$$\Theta_t = -\operatorname{div}\vec{\phi} + Q$$

$$\boxed{(c\rho u)_t = \operatorname{div}(K\,Du) + Q}$$

spatial gradient

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

$$\int_R \left[(c\rho u)_t - \operatorname{div}(K\,Du) - Q \right] = 0$$

weak version

[redacted]

$$\operatorname{div}(K\,Du) = \underset{\uparrow\ \text{const.}}{K}\ \underline{\underline{\operatorname{div} Du}}$$

$$\operatorname{div} \mathbb{V} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \qquad \mathbb{V} = (v_1, v_2, \ldots, v_n)$$

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

$$\operatorname{div} Du = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots$$

$$= \Delta u \text{ (Laplacian)}$$

[redacted]

$$(c\rho u)_t = \operatorname{div}(K D u) + Q$$

$$u_t = k\Delta u + f$$

$$Lu = u_t - k\Delta u \quad \longleftarrow \text{ heat operator}$$

$Lu = f$ linear 2^{nd} order parabolic PDE.