MATH 4581 Lecture 27 November 30, 2021

<u>Last time</u>: Fundamental Modes of the circular drum

$$u\, \cancel{u}_{jk} = C_{jk}\left(a_{jk}\cos j\theta + b_{jk}\sin j\theta\right) J_j\left(z_{jk}\frac{r}{a}\right)$$

$J_j = $ "J_ν" $j^{\underline{th}}$ order Bessel function of the first kind.

$z_{jk} = k^{th}$ positive zero of J_j

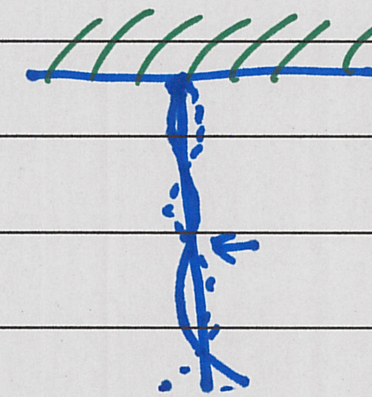
wave form.

$$C_{jk} = \alpha_{jk}\cos\left(\sigma\frac{z_{jk}}{a}t\right) + \beta_{jk}\sin\left(\sigma\frac{z_{jk}}{a}t\right)$$

oscillation

[redacted]

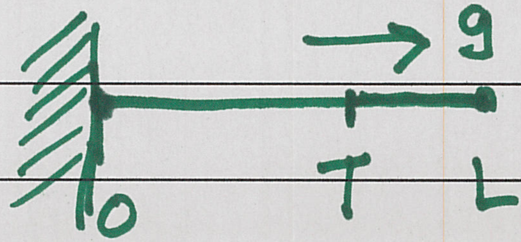
Today: (1) Transverse oscillations of a hanging chain.



(2) Numerics of Bessel functions and Bessel-like functions.

Model

$$\boxed{\rho u_{tt} = (T u_x)_x}$$



$$T(x) = \rho g (L-x)$$

[redacted]

$\rho u_{tt} = (T u_x)_x \leftarrow$ Assume $T = T(x)$

e.g. $T = \rho g(L-x)$

$u = A(x)B(t)$

$\rho A B'' = B(TA')'$

$\frac{B''}{B} = \frac{(TA')'}{\rho A} = -\lambda = -\mu^2$

oscillation

$TA'' + T'A' + \mu^2 \rho A = 0$

$g(L-x)A'' - gA' + \mu^2 A = 0$

[redacted]

$$g\,\underline{(L-x)}\,A'' - g\,A' + \mu^2 A = 0$$

singular point at $x = L$.

$$\xi = L - x, \quad \begin{cases} f(\xi) = A(L-\xi) \\ f' = -A' \\ f'' = A \end{cases}$$

$$g\,\xi f'' + g f' + \mu^2 f = 0$$

singular at $\xi = 0$

Bessel ODE

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

[redacted]

$$? \begin{cases} f'' + \dfrac{1}{\xi} f' + \dfrac{\mu^2}{g\xi} f = 0 \\[2ex] y'' + \dfrac{1}{x} y' + \left(1 - \dfrac{\nu^2}{x^2}\right) y = 0 \end{cases}$$

<u>THEOREM</u>: If y satisfies the Bessel ODE of order ν, then $f(\xi) = \xi^p\, y(a\xi^q)$ satisfies

$$f'' + \frac{1-2p}{\xi} f' + \left(\frac{p^2 - q^2\nu^2}{\xi^2} + a^2 q^2 \xi^{2q-2}\right) f = 0.$$

Bessel-type ODE

<u>AND</u> $f_\nu = \xi^p J_\nu(a\xi^q)$ and $g_\nu = \xi^p Y_\nu(a\xi^q)$ is a basis of solutions on $\xi > 0$.

[redacted]

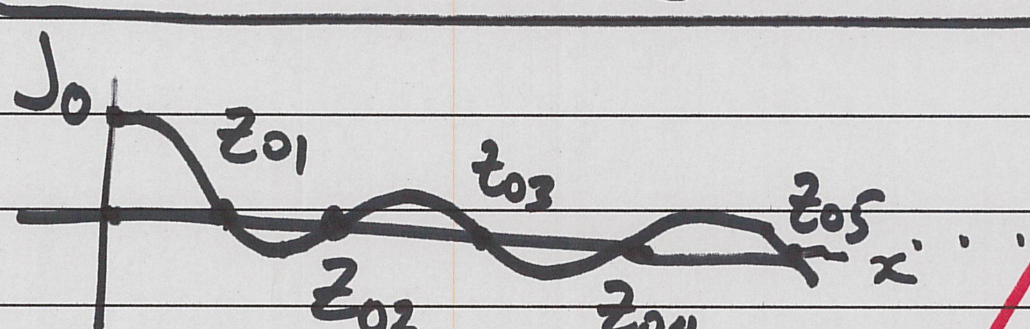
$$f'' + \frac{1}{\xi} f' + \frac{\mu^2}{g\xi} f = 0$$

$$f'' + \frac{1-2p}{\xi} f' + \left(\frac{p^2 - q^2 \nu^2}{\xi^2} + a^2 q^2 \xi^{2q-2} \right) f = 0$$

$$p = 0, \quad \nu = 0, \quad q = \frac{1}{2}, \quad a = \frac{2\mu}{\sqrt{g}}.$$

$$\boxed{f(\xi) = J_0\left(\frac{2\mu}{\sqrt{g}} \xi^{1/2} \right)} \leftarrow \text{wave form(s)}$$

Need $J_0\left(2\mu\sqrt{\frac{L}{g}}\right) = 0$



$$A(x) = J_0\left(2\mu \sqrt{\frac{L-x}{g}} \right)$$

[redacted]

$$J_0\left(2\mu\sqrt{\frac{L}{g}}\right)=0$$

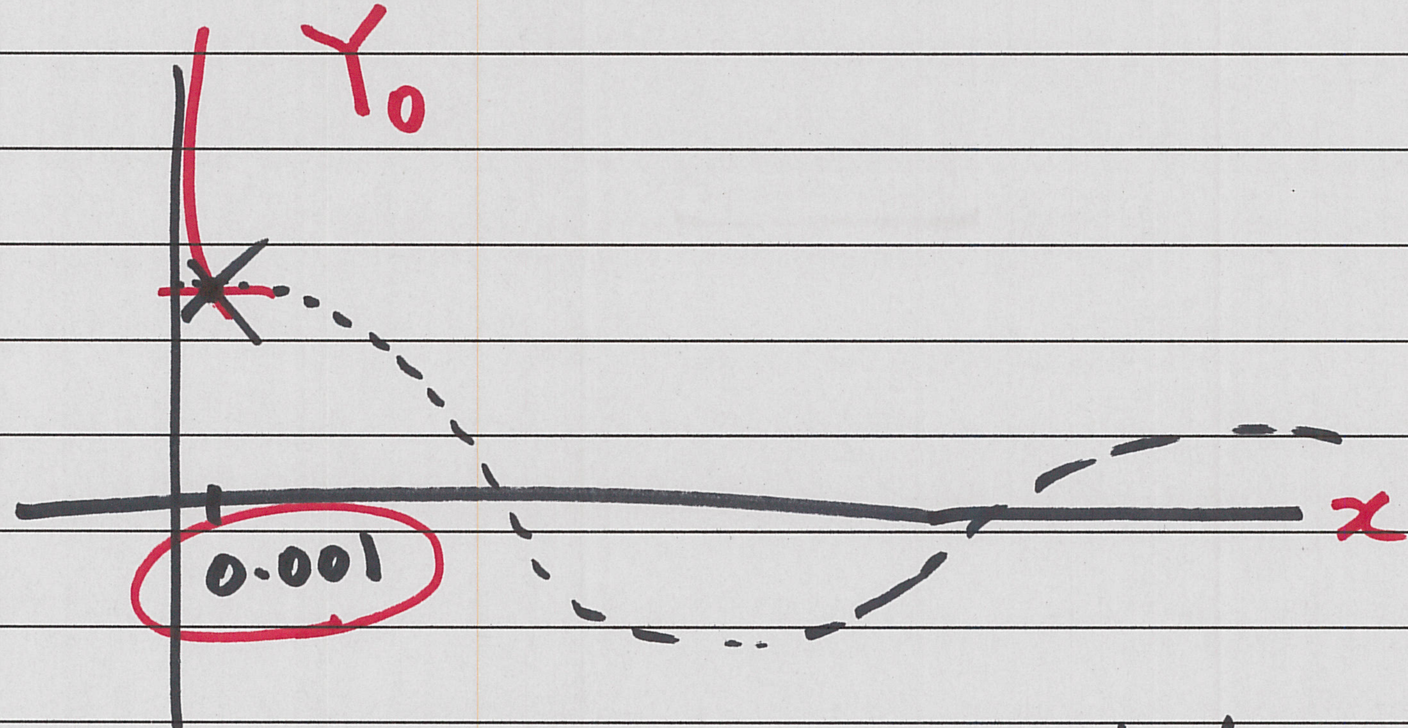
$$\Longrightarrow\quad 2\mu_j\sqrt{\frac{L}{g}} = Z_{0j}$$

$$\mu_j = \frac{Z_{0j}}{2}\sqrt{\frac{g}{L}}$$

<u>Wave form(s)</u> $A(x) = J_0\left(Z_{0j}\sqrt{1-\frac{x}{L}}\right)$

$$u_j = \underbrace{\cos\left(\frac{Z_{0j}}{2}\sqrt{\frac{g}{L}}\,t\right)}_{\rho?} J_0\left(Z_{0j}\sqrt{1-\frac{x}{L}}\right.$$

[redacted]



Guess: If $p = -1$, what happens with $y(0.000001)$? $+\infty$

If $p = +1$, $y(0.000001) = ?$ $-\infty$

Plot $y(0.000001)$ as a function of p [redacted]