MATH 4581 Lecture 24 Tuesday November 16, 2021

- HANGING SLINKY.

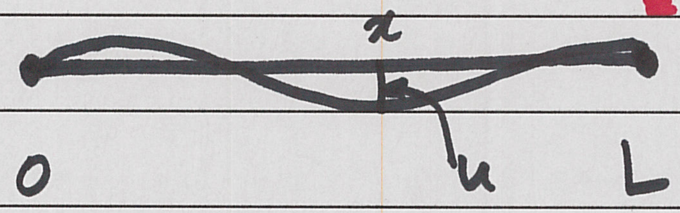
- Sturm-Liouville Theory (Ch. 5)
  - Bessel functions

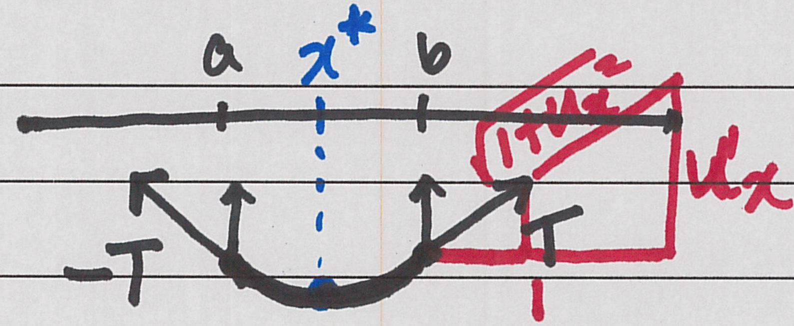
- transverse oscillations of hanging chain

[redacted]

Transverse ~~Oscillations~~ vibrations (The wave eqn. according to Haberman et al.)

$u = (x, t)$





"derivation"

Mass $M \approx \rho(b-a)$

ρ = density $[\rho] = \frac{M}{L}$

T = tension

$$F(b) = T \frac{u_x(b)}{\sqrt{1+u_x^2(b)}} \approx T u_x(b)$$

Newton's 2nd Law

$$\rho(b-a)\,u_{tt}(x^*) \approx T u_x(b) - T u_x(a)$$

[redacted]

$$\rho\, u_{tt}(x^*) \simeq \frac{T\,u_x(b) - T\,u_x(a)}{b-a}$$

Take the limit as $b, a \to x \in (0, L)$

$$\rho\, u_{tt}(\underset{\wedge}{x}) = T\,u_{xx}(\underset{\wedge}{x})$$

Better: $\rho\, u_{tt}(x^*, t) \simeq \dfrac{T\,u_x(b,t) - T\,u_x(a,t)}{b-a}$

$$\rightarrow \boxed{\rho\, u_{tt} = T\,u_{xx}}$$

$$\boxed{u_{tt} = \sigma^2 \Delta u}$$

[redacted]

Chapter 5 Sturm-Liouville Theory

$$\begin{cases} A'' = -\lambda A \\ A(0) = 0 = A(L) \end{cases}$$ Sturm-Liouville (eigenvalue) problem.

There is a sequence of eigenvalues

$$\lambda_1 = \frac{\pi^2}{L^2},\ \lambda_2 = \frac{4\pi^2}{L^2},\ \lambda_3 = \frac{9\pi^2}{L^2}, \dots,\ \lambda_j = \frac{j^2\pi^2}{L^2}$$

with corresponding eigenfunctions /solutions

$$A_1 = \sin\frac{\pi}{L}x,\ A_2 = \sin\frac{2\pi}{L}x, \dots,\ A_j = \sin\frac{j\pi}{L}x.$$

Any L^2 function f can be expanded as a series

$$f = \sum_{j=1}^{\infty} a_j A_j(x)$$

and the coefficients are (relatively) eas [redacted]
to find because...

$$\frac{2}{L}\int_0^L A_i(x)A_j(x)\,dx = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

(orthogonality)

Sturm-Liouville Theory says these same things hold (for different eigenfunctions) for

$$\boxed{(pA')' + (q+\lambda r)A = 0 \quad \text{on } (a,b)}$$

$p, q, r \in C^0[a,b]$, $\underline{\underline{p, r > 0}}$

"regular"

Regular Sturm-Liouville Problem:

$$\begin{cases} (pA')' + (q+\lambda r)A = 0 & \text{on } (a,b) \\ c_{11}A(a) + c_{12}A'(a) = 0 \\ c_{21}A(b) + c_{22}A'(b) = 0 \end{cases}$$

← typical (periodic are also okay)

↑ homogeneous

$p = p(x),\ q = q(x),\ r = r(x)$, $p, q, r \in C^0[a,b]$

$p, r > 0$ (regular)

$A : [a,b] \to \mathbb{C}$. typically A can be complex valued.

There exists a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lambda_j \nearrow +\infty$ of eigenvalues with corresponding $A_1, A_2, A_3, \ldots$

[redacted]

$A_1, A_2, \ldots$

determining unique one-dimensional subspaces of L^2, i.e.,

$$\Sigma_j = \{ a A_j : a \in R \}$$

is the solution set / eigenspace

AND any $f \in L^2$ can be written uniquely as

$$f = \sum_{j=1}^{\infty} a_j A_j$$ (converging in L^2, uniformly if $f \in C^0[a,b]$ etc.)

and $$\frac{1}{c_i} \int_a^b A_i(x) A_j(x) r(x) \, dx = \delta_{ij}$$ ↑ weight

where $c_i = \int A_i^2 r = \|A_i\|^2_{L^2_r}$.

[redacted]

<u>7.2-7.7</u> vibrating drum head

$$\begin{cases} u_{tt} = \sigma^2 \Delta u \quad \text{on } \mathcal{U} \times (0,\infty), \quad \mathcal{U} \subseteq \mathbb{R}^2 \\ u\big|_{\partial \mathcal{U}} = 0, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \end{cases}$$

<u>Separation of variables</u>: $u = A(t)\,B(x)$

$= A(t)\,X(x)\,Y(y)$

$$A''B = \sigma^2 A \Delta B$$

$$\frac{1}{\sigma^2}\frac{A''}{A} = \frac{\Delta B}{B} = \lambda$$

[redacted]

<u>time problem</u> : $A'' = -\lambda\sigma^2 A$

<u>spatial problem</u> : $\begin{cases} \Delta B = -\lambda B \\ B\big|_{\partial U} = 0 \end{cases}$ ← good boundary values.

<u>§7.4</u> The eigenvalue problem for the Laplacian.

$$\begin{cases} \Delta B + \lambda B = 0 \quad \text{on } U \subseteq \mathbb{R}^n \\ c_1 B\big|_{\partial U} + c_2 D_n u\big|_{\partial U} = 0 \end{cases}$$

Take B_{jk} orthogonal in L^2

$\int_U B_{jk} B_{jl}$ [redacted]

There exists a sequence $\lambda_1 < \lambda_2 < \cdots \lambda_j \to \infty$ [redacted]

corresponding to each λ_j is a finite [redacted]

dim'l eigenspace $\Sigma_j = \operatorname{span}\{B_{j1}, B_{j2}, \ldots$ [redacted]

$f(x) \overset{L^2}{=} \sum_{j,k} a_{jk} B_{jk}$

Think $\mathcal{U}$ = rectangle $(0,L)\times(0,M)$

$$B_{jk} = \sin\frac{j\pi x}{L}\,\sin\frac{k\pi y}{M}$$

$$\Delta B_{jk} = \underbrace{-\left(\frac{j^2\pi^2}{L^2}+\frac{k^2\pi^2}{M^2}\right)}_{-\lambda_{jk}} B_{jk}$$

$$\frac{j^2\pi^2}{L^2}+\frac{k^2\pi^2}{M^2}$$

Generalizes: $\operatorname{div}(p\nabla B) + (q+\lambda r)B = 0$

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