MATH 4581 Lecture 22 Tuesday November 9, 2021

* Assignment 5 Problem 10

* LAST TIME d'Alembert's solution of

$$\Box u = 0$$

$$\Box u = u_{tt} - \sigma^2 u_{xx} = (u_t + \sigma u_x)_t - \sigma(u_t + \sigma u_x)_x$$

or

$$= \underbrace{(u_t - \sigma u_x)}_{= W}{}_t + \sigma (u_t - \sigma u_x)_x$$

[redacted]

$$(u_t - \sigma u_x)_t + \sigma \underbrace{(u_t - \sigma u_x)}_{Mu}{}_x = 0$$

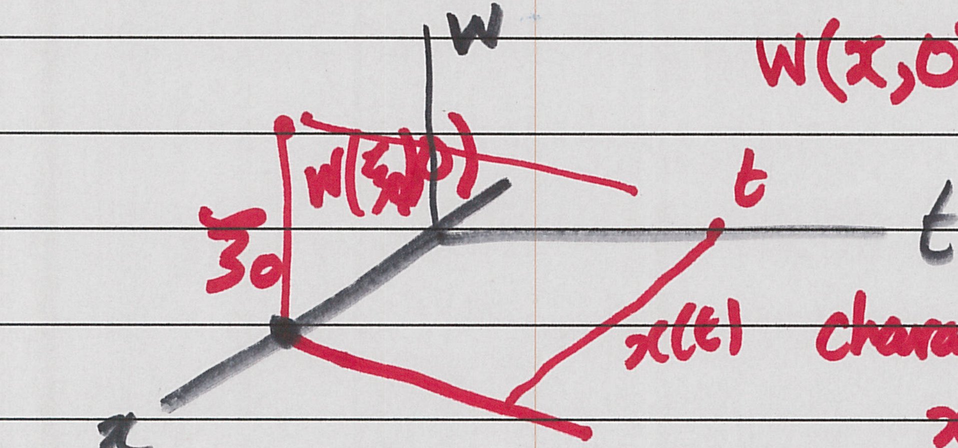
$$LW = W_t + \sigma W_x$$

$LW = 0$

$W_t + \sigma W_x = 0$

Cauchy Data along $t = 0$. (non-characteristic curve)

$W(x,0) = \underline{V_0(x) - \sigma U_0'(x)}$




characteristic curve

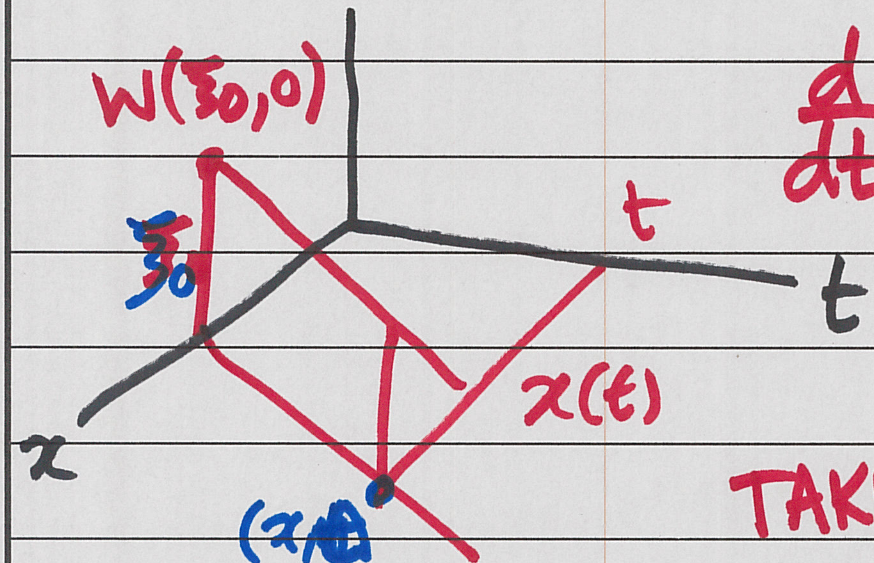
$x = x(t)$

Method of characteristics

[redacted]

Method of Characteristics

$$\begin{cases} W_t + \sigma W_x = 0 \quad (*) \\ W(x,0) = V_0(x) - \sigma U_0'(x) \end{cases} \leftarrow \text{initial/Cauchy data}$$




$$\frac{d}{dt} W(x(t),t)$$

$$= W_x \, x' + W_t$$

← compare to PDE $(*)$

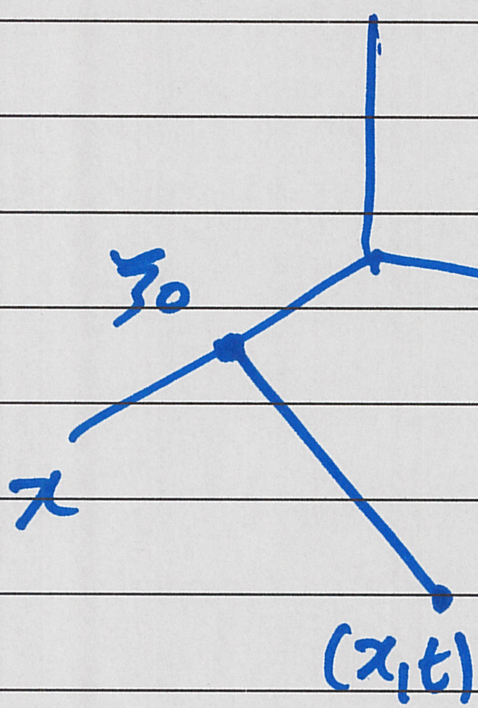
TAKE $x' = \sigma$

$$x(t) = \xi_0 + \sigma t$$

Conclusion : $W(x(t),t) \equiv W(\xi_0, 0)$

$$W(\xi_0 + \sigma t, t) = \underline{V_0(\xi_0) - \sigma U_0'(\xi_0)}$$

[redacted]




$x(t) = \xi_0 + \sigma t$

$\underline{\underline{Q}}$: Is there a $\xi_0$ (starting point) for which $\xi_0 + \sigma t = x$ ?

$$W(\xi_0 + \sigma t, t) = V_0(\xi_0) - \sigma U_0'(\xi_0)$$

$$W(x,t) = \underline{V_0(x - \sigma t) - \sigma U_0'(x-\sigma t}$$ [redacted]

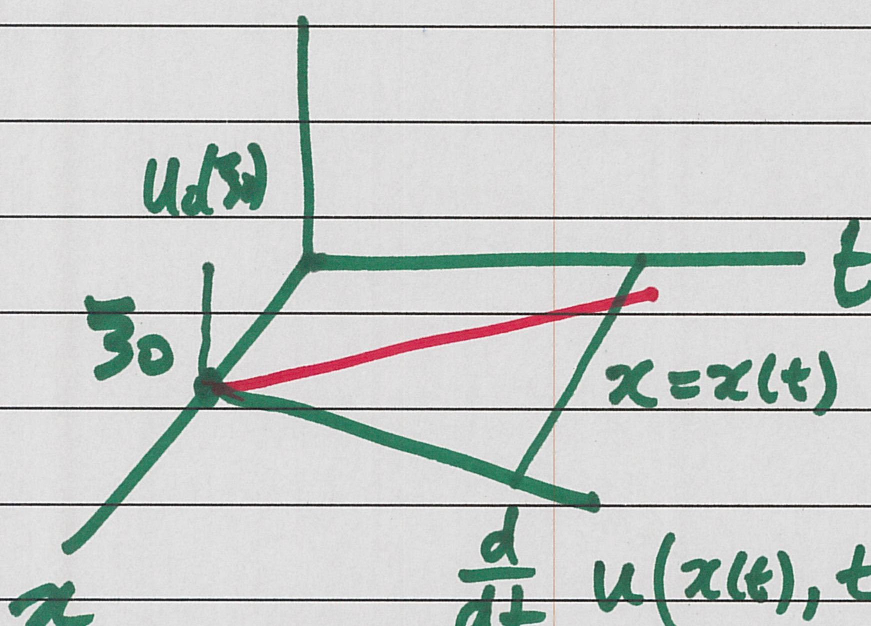
$\boxed{U_{tt} = \sigma^2 U_{xx}}$ $\quad W_t = \underline{\underline{U_t - \sigma U_x}}$

Mu

# Second $1^{st}$ order PDE

$$\begin{cases} u_t - \sigma u_x = W = \underbrace{V_0(x-\sigma t) - \sigma u_0'(x-\sigma t)}_{\text{given inhomogeneity}} \\ u(x,0) = u_0(x) \end{cases}$$

$$\begin{cases} u_{tt} = \sigma u_{xx} \\ u(x,0) = u_0 \\ u_t(x,0) = V_0 \end{cases}$$




$$\frac{d}{dt} u(x(t),t) = u_x x' + u_t$$

TAKE $x' = -\sigma$

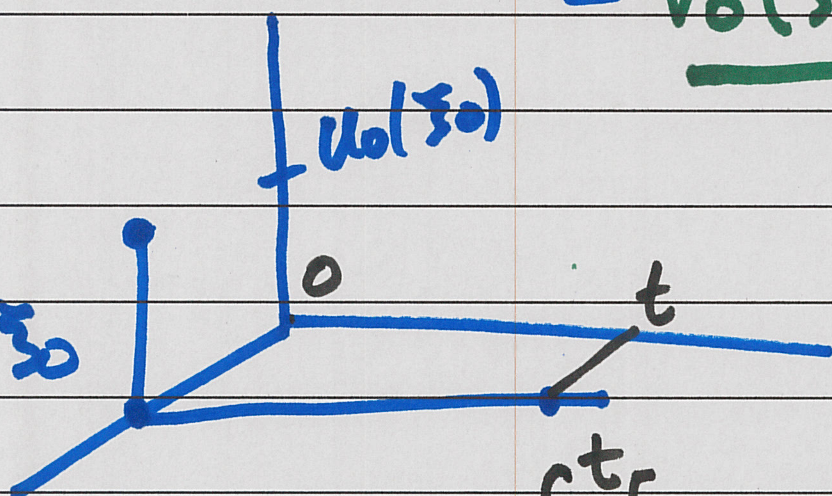
[redacted]

$$\begin{cases} u_t - \sigma u_x = v_0(x-\sigma t) - \sigma\, u_0'(x-\sigma t) \\ u(x,0) = u_0(x) \end{cases}$$

<u>Characteristic</u> : $x(t) = \xi_0 - \sigma t$

$$\frac{d}{dt}\, u(x(t),t) = u_x(-\sigma) + u_t$$

$$= \underbrace{v_0(\xi_0 - 2\sigma t) - \sigma\, u_0'(\xi_0 - 2\sigma t)}_{\text{given function of } t}$$




$$u(\xi_0 - \sigma t, t) - u(\xi_0, 0) = \int_0^t \left[ v_0(\xi_0 - 2\sigma\tau) - \sigma\, u_0'(\xi_0 - 2\sigma\tau) \right] d\tau$$

[redacted]

$$u(\xi_0-\sigma t,t) - u(\xi_0,0) = \int_0^t \left[ v_0(\xi_0-2\sigma\tau) - \sigma u_0'(\xi_0-2\sigma\tau) \right] d\tau$$

$$= \int_0^t v_0(\xi_0-2\sigma\tau)\, d\tau - \sigma \int_0^t \underline{u_0'(\xi_0-2\sigma\tau)}\, d\tau$$

$$= \int_0^t v_0(\xi_0-2\sigma\tau)\, d\tau + \frac{1}{2}\, u_0(\xi_0-2\sigma\tau)\Big|_{\tau=0}^{t}$$

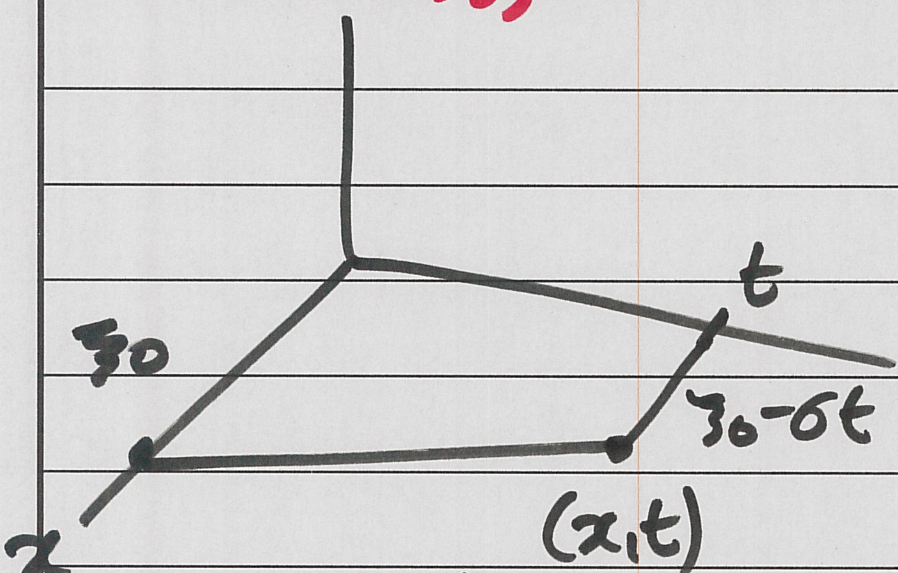
<u>side computation:</u> $\frac{d}{d\tau} u_0(\xi_0-2\sigma\tau) = -2\sigma u_0'(\xi_0-2\sigma\tau)$

$$u(\xi_0-\sigma t,t) - u_0(\xi_0) = \int_0^t v_0(\xi_0-2\sigma\tau)\, d\tau + \frac{1}{2}\left[u_0(\xi_0-2\sigma t) - u_0(\xi_0)\right]$$

[redacted]

$$u(\xi_0 - \sigma t, t) = \int_0^t v_0(\xi_0 - 2\sigma\tau)\,d\tau + \frac{1}{2}\left[u_0(\xi_0 - 2\sigma t) - u_0(\xi_0)\right]$$

$$-u_0(\xi_0)$$




choose $\xi_0$ so that

$$\xi_0 - \sigma t = x$$

(?)

$$u(x,t) = \int_0^t v_0(x + \sigma t - 2\sigma\tau)\,d\tau + \frac{1}{2}\left[u_0(x - \sigma t) \,\cancel{+}\, - \underline{\underline{u_0(x+\sigma t)}}\right]$$

$$+ \underline{\underline{u_0(\xi_0)}}$$

$$= \int_0^t v_0(x + \sigma t - 2\sigma\tau)\,d\tau + \frac{1}{2}\left[u_0(x - \sigma t) + \underline{u_0(x + \sigma t)}\right]$$

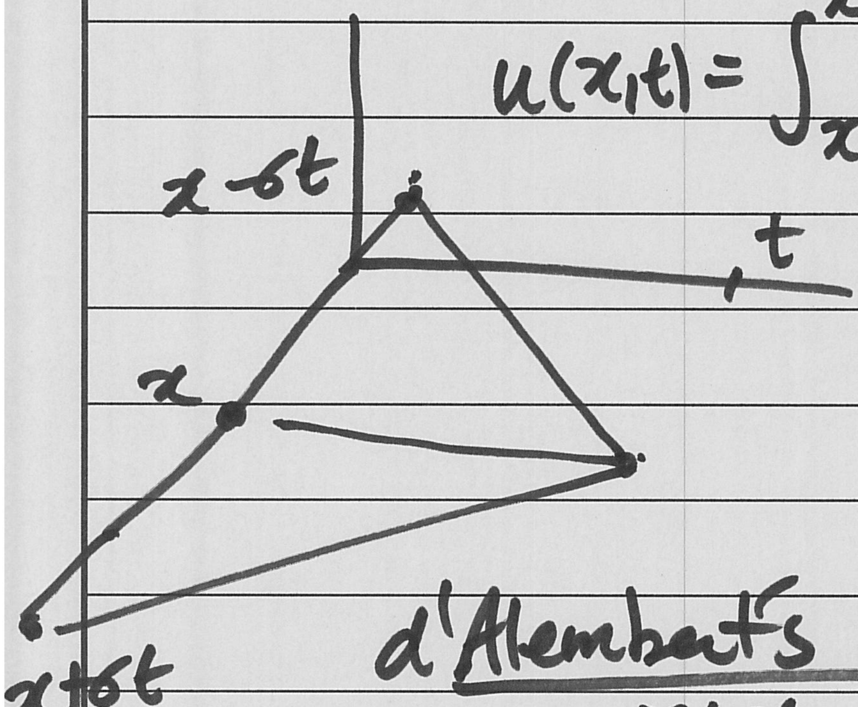
[redacted]

$$u(x,t) = \int_0^t \underline{V_0(x+\sigma t - 2\sigma\tau)\, d\tau} + \frac{1}{2}\left[u_0(x-\sigma t) + u_0(x+\sigma t)\right]$$

<u>change variables</u> $\xi = x + \sigma t - 2\sigma\tau \quad , \quad d\xi = -2\sigma\, d\tau$

$$u(x,t) = \int_{x+\sigma t}^{x-\sigma t} V_0(\xi)\left(-\frac{1}{2\sigma}\right) d\xi$$

$$+ \frac{1}{2}\left[u_0(x-\sigma t) + u_0(x+\sigma t)\right]$$

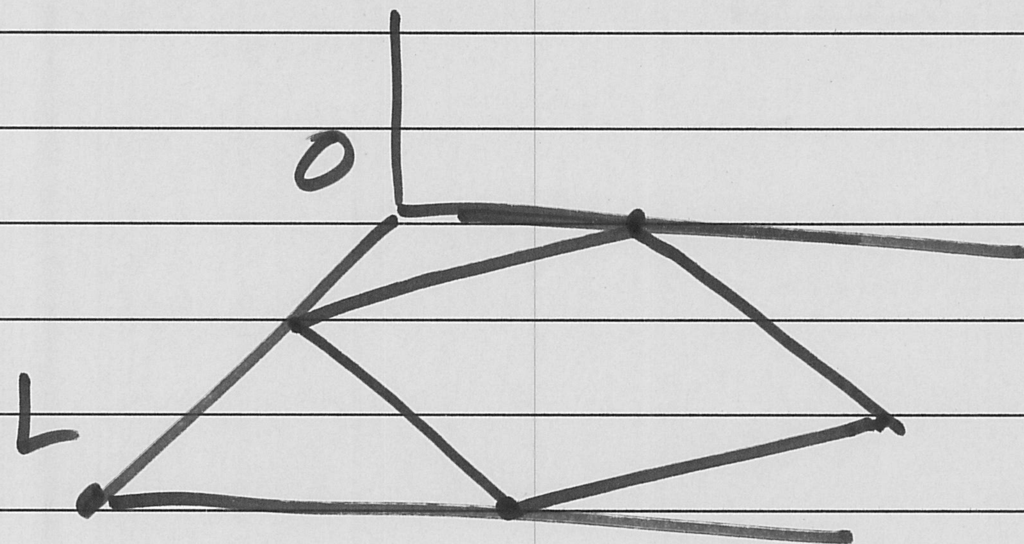



d'<u>Alembert's Solution</u>:

$$u(x,t) = \frac{1}{2\sigma}\int_{x-\sigma t}^{x+\sigma t} V_0(\xi)\, d\xi + \frac{1}{2}\left[u_0(x-\sigma t) + u_0(x+\sigma t)\right]$$

[redacted]

With boundaries

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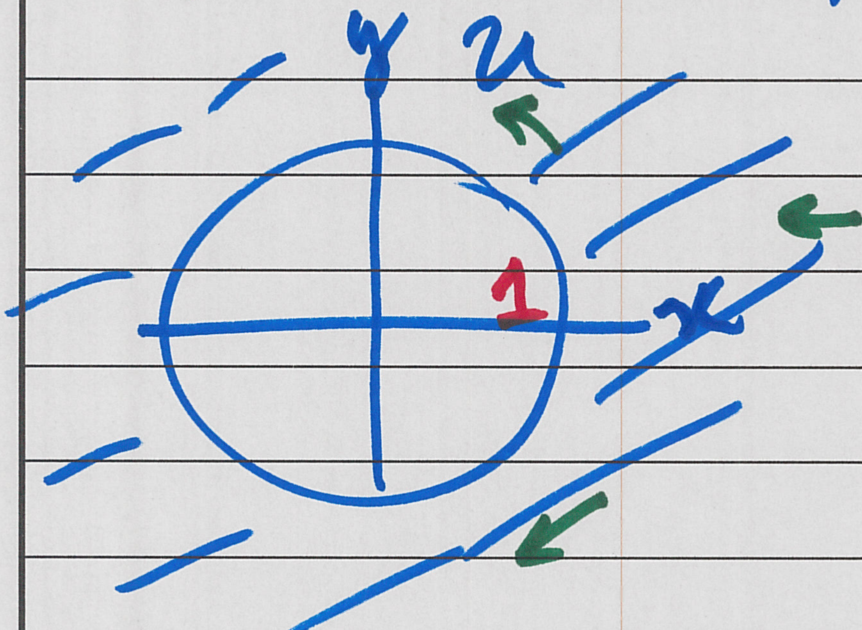
## Streamlines  Problem 10  Assignment 5

$\psi : \mathcal{U} \longrightarrow \mathbb{R}$ , $\mathcal{U} = \mathbb{R}^2 \setminus B_1(0)$





$$\psi(x,y) = -\alpha \ln\sqrt{x^2+y^2} - y\left(1 - \frac{1}{x^2+y^2}\right)$$

$$\psi(r,\theta) = -\alpha \ln r - r\sin\theta\left(1 - \frac{1}{r^2}\right)$$
$$= -\alpha \ln r - \sin\theta\left(r - \frac{1}{r}\right)$$

[redacted]

Level sets for $\psi$

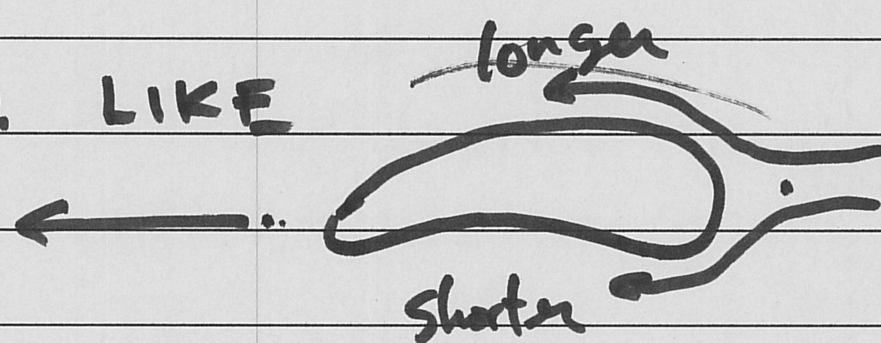
$$\psi = -\alpha \ln \sqrt{x^2+y^2} - y\left(1 - \frac{1}{x^2+y^2}\right)$$

$$L_h = \{(x,y) \in \mathbb{R}^2 : \psi(x,y) = h\}$$

← Level curve for $\psi$.

$$\mathcal{L}_h = \{(r,\theta) \in \Sigma_1 : \Psi(r,\theta) = h\}$$

IDEA: LIKE

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