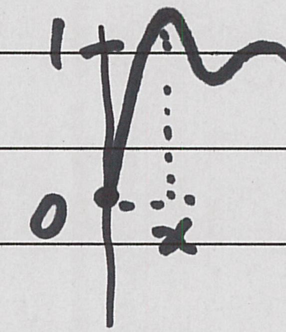
MATH 4581 Lecture 21, Thursday Nov. 4, 2021

- hanging slinky
- Gibb's phenomenon (Assignment 5 Problem 6)
  - trig identity
- river crossing (Assignment 5 Problem 7)
- d'Alembert's solution of the wave equation
  - the wave equation on all of $\mathbb{R}$

[redacted]

$$\sum_{j=0}^{N} \cos[(2j+1)x] = \frac{\sin[(2N+1)x]}{2\sin x} \quad (*)$$



Gibb's Phenom.

Max at $x = \frac{\pi}{2(N+1)}$

Proof of identity by induction:

base case: $N=0$.

$$\cos(x) \stackrel{?}{=} \frac{\sin(2x)}{2\sin x} \checkmark$$

It's time to eat, Grandma.

(commas save lives)

Assume (*) is true for $N=K$

then show (*) holds when $N=K+1$.

← INDUCTIVE STEP.

[redacted]

Assume $\displaystyle\sum_{j=1}^{K} \cos[(2j+1)x] = \frac{\sin[2(K+1)x]}{2\sin x}$

Look At

$$\sum_{j=1}^{K+1} \cos[(2j+1)x] = \sum_{j=1}^{K} \cos[(2j+1)x] + \cos[(2K+3)x]$$

To show: $\displaystyle\frac{\sin[2(K+1)x]}{2\sin x} + \cos[(2K+3)x] = \frac{\sin[2(K+2)x]}{2\sin x}$

$$\sin[(2K+2)x] + 2\sin x \cos[(2K+3)x]$$

$$\stackrel{?}{=} \sin[(2K+4)x]$$

[redacted]

To show:

$$\sin[(2k+2)x] + 2\underline{\sin x \cos[(2k+3)x]} = \sin[(2k+4)x]$$

$\underline{\text{use}}$: $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\sin(A-B) = \sin A \cos B - \sin B \cos A$

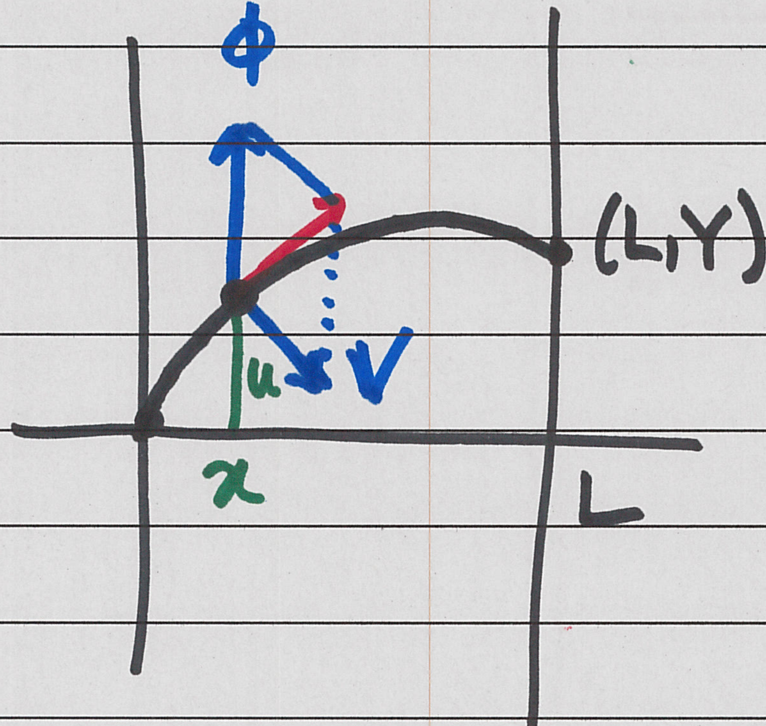
$$\sin[(2k+3)x - x] = \sin[(2k+3)x]\cos x$$
$$\underline{- \sin x \cos[(2k+3)x]}$$

$$\sin[(2k+3)x]\cos x + \sin x \cos[(2k+3)x]$$

$$= \sin[(2k+4)x].$$

[redacted]

Last time



$W = (v_1, v_2)$

$$\underset{\overset{\shortparallel}{\frac{dx}{dt}}}{x'}\,(1, \underset{\overset{\shortparallel}{\frac{du}{dx}}}{u'}) = (v_1, v_2 + \phi)$$

$$\underset{\uparrow \text{crossing time}}{T} = \int_0^L \frac{1}{v_1}\,dx$$

(i) What is the argument of v_1?

(ii) What are we going to minimize T over?

$\underline{\underline{v_1 = v_1(x)}}$ $\qquad \uparrow\ T = T[u].$

[redacted]

IDEA: Express v_1 in terms of u (and u'...)

$$A = \{ u \in C^1[0,L] : u(0)=0,\ u(L)=Y,\ \ldots \}$$

$\leftarrow v > \max_{[0,L]} \phi$

$$\dot{x}(1, u') = (v_1, v_2 + \phi)$$

Note: v_1 and v_2 are related $|\mathbb{V}| = v$ const.

$$\boxed{v_1^2 + v_2^2 = v^2} \text{ (given constant).}$$

$$u' = \frac{v_2 + \phi}{v_1} = \frac{\phi}{v_1} \pm \underbrace{\sqrt{\left(\frac{v}{v_1}\right)^2 - 1}}_{\frac{v_2}{v_1}\ ?}$$

[redacted]

$$u' = \frac{\phi}{v_1} \pm \sqrt{\left(\frac{v}{v_1}\right)^2 - 1} \quad \leftarrow \text{solve for } v_1$$

$$\boxed{u'^2 - \frac{2\phi}{v_1} u' + \frac{\phi^2}{v_1^2} = \frac{v^2}{v_1^2} - 1}$$

↖ really good because sign ± doesn't matter.

$$\frac{v^2 - \phi^2}{v_1^2} + 2\phi u' \frac{1}{v_1} - (1 + u'^2) = 0$$

$$\frac{1}{v_1} = \frac{-2\phi u' \pm \sqrt{4\phi^2 u'^2 + 4(v^2 - \phi^2)\cdot(1+u'^2)}}{2(v^2 - \phi^2)}$$

[redacted]

$$\frac{1}{V_1} = -\frac{\phi u'}{V^2-\phi^2} + \frac{1}{V^2-\phi^2}\sqrt{\underline{\phi^2 u'^2} + (V^2-\phi^2)(1+u'^2)}$$

$$= -\frac{\phi}{V^2-\phi^2}\, u' + \frac{1}{V^2-\phi^2}\sqrt{V^2(1+u'^2) - \phi^2}$$

$$= -\frac{\phi}{V^2-\phi^2}\, u' + \sqrt{\frac{V^2}{(V^2-\phi^2)^2}u'^2 + \frac{1}{V^2-\phi^2}}$$

$$= -\phi\psi\, u' + \sqrt{(V\psi)^2 u'^2 + \psi}$$

$$\psi = \frac{1}{V^2-\phi^2}$$

[redacted]

Minimize:

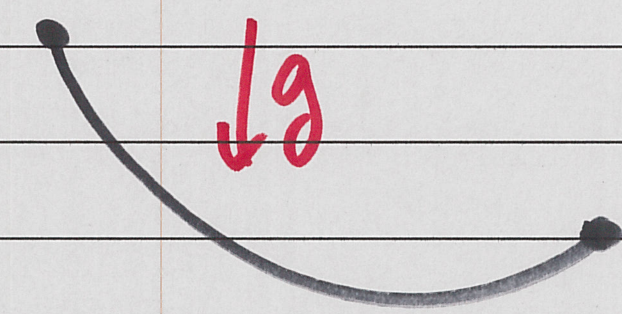
$$T[u] = \int_0^L \left[\sqrt{(V\psi)^2 u'^2 + \psi} - \phi\psi u' \right] dx$$

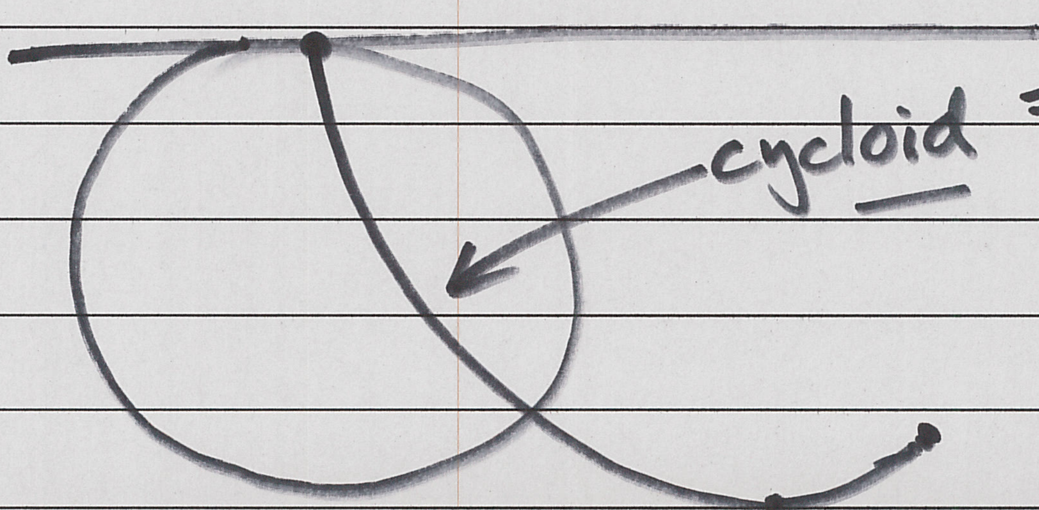
Find $\delta T_u[\eta]$ ...

[redacted]

Brachisto chrone

shortest time





cycloid = path of a point on a rolling circle.

[redacted]

Wave Equation on all of $\mathbb{R}$

$$u_{tt} = \sigma^2 u_{xx}$$

The wave operator $\quad \Box u = u_{tt} - \sigma^2 u_{xx}$

d'Alembert operator

<u>factors</u> : $(u_t - \sigma u_x)_t + \sigma(u_t - \sigma u_x)_x$

$= (u_t + \sigma u_x)_t - \sigma(u_t + \sigma u_x)_x$

$$\begin{cases} Lu = u_t + \sigma u_x \\ Mu = u_t - \sigma u_x \end{cases} \qquad \begin{aligned} \Box u &= L \circ M u \\ &= M \circ L u \end{aligned}$$

[redacted]

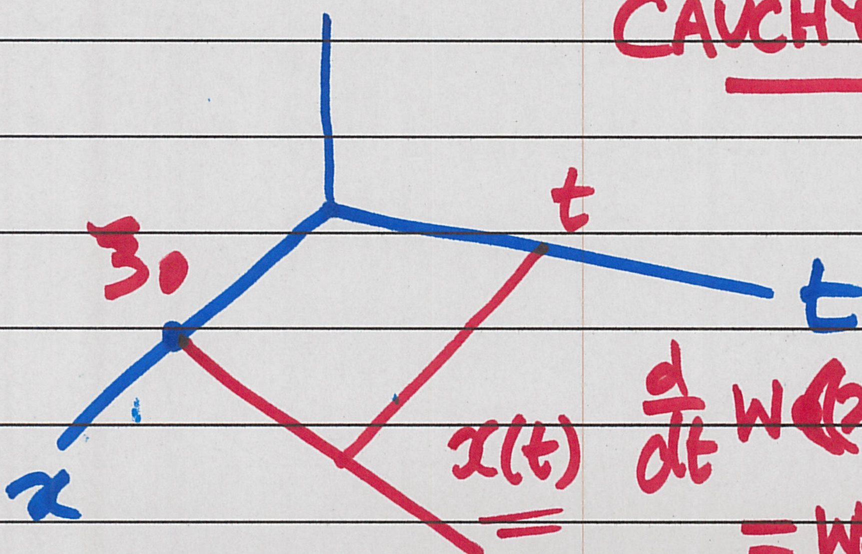
1^{st} order PDE AND THE method of characteristics

$$\underbrace{(u_t - \sigma u_x)}_{Mu = W}{}_t + \sigma(u_t - \sigma u_x)_x = 0$$

initial conditions $\begin{cases} u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases}$

$W_t + \sigma W_x = 0 \leftarrow 1^{st}$ order PDE

CAUCHY DATA : $W(x,0) = v_0(x) - \sigma u_0'(x)$



$$\frac{d}{dt} W(x(t),t) = W_x \cdot x' + W_t$$

[redacted]