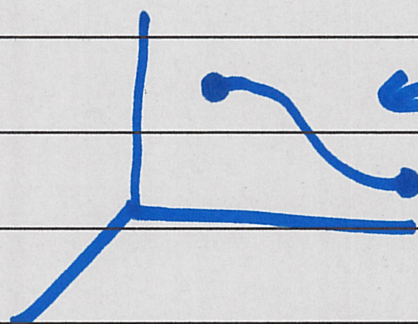
MATH 4581 Lecture 20 Nov. 2, 2021

Last time: Derivation of $W_{tt} = \sigma^2 W_{xx}$

$$(MA) = \int \rho \, W_{tt} = F$$

$$mv_2 - mv_1 = F \Delta t \quad (2^{\text{nd}} \text{ derivation})$$

Hamilton's Stationary Action (3^{rd} derivation)



← something moves

$X = X(t)$ on $[0,T]$

[redacted]

Action $[X] = E - K$

total potential energy

Point mass motions (Newtonian Mechanics)

$$\text{Action}[\underline{x}] = \int_0^T \underline{\Phi}(\underline{x}(t))\,dt - \frac{1}{2}\int_0^T M|\underline{x}'|^2\,dt$$

potential function

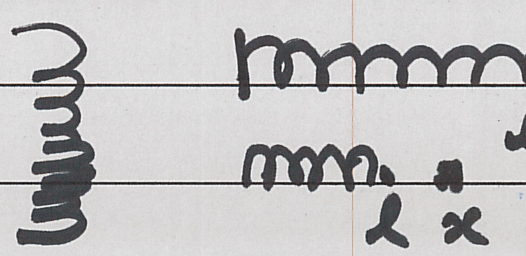
TOTAL KINETIC ENERGY.

<u>Hanging Slinky</u>

<u>Force balance</u>

$$\int_{\ell}^{\ell'} [k(\underline{x-\ell})]\,dx$$

<u>Alternative: Potential Energy.</u>



$F = -k(\ell' - \ell)$

$E = \frac{1}{2}k(\ell' - \ell)^2$

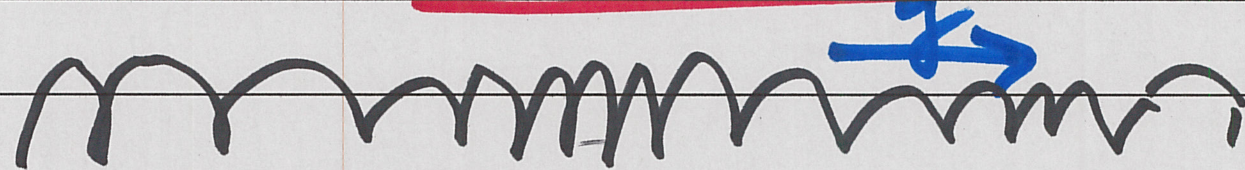
$F = -k(x - \ell)$

[redacted]

unit length potential

$$E_{hom.} = \frac{1}{2} k (l' - l)^2 = \frac{1}{l^2}\int_0^l \underbrace{\frac{\epsilon}{2}(l'-l)^2}_{=\frac{\epsilon l^2}{2}(w'-1)^2} dx$$

$$E_{inhom.} \overset{?}{=} \cancel{l^2}\int_0^l \frac{\epsilon}{2}(w'-1)^2 dx$$



$w' - 1$

$$\delta E_{inhom.} = 0 \quad ?$$

$$\frac{d}{d\alpha}\int_0^l (w' + \alpha\phi' - 1)^2 dx$$

$$2\int_0^l (w' + \alpha\phi' - 1)\phi' dx$$

$$\int_0^l (w'-1)\phi' dx = 0$$

for all ϕ.

$$\int_0^l w''\phi \, dx = 0$$

[redacted]

$\underline{w'' = 0}$ $\quad$ w is $\underline{\text{affine}}$

Not the correct conclusion.

Left out gravitational potential energy.

$$\boxed{w'' = -\frac{\rho g}{\epsilon}}$$

agrees with force balance
but gives more solutions.

w is QUADRATIC

$$w = -\frac{\rho g}{2\epsilon}x^2 + \left(\frac{\rho g}{\epsilon}L + 1\right)x$$

[redacted]

Let's try to check it!

$$W(x) = -\frac{\rho g}{2\varepsilon}x^2 + \left(\frac{\rho g}{\varepsilon}L + 1\right)x$$

Mass = 212 g = 0.212 kg.
Length L = 55.4 mm = 0.0554 m.

$$\rho = \frac{0.212}{0.0554}\ \mathrm{kg/m}$$

84 coils ← coils tighten when it hangs

heights: 0, $3\frac{5}{8}$, $5\frac{13}{16}$, $8\frac{5}{8}$, 11,
$13\frac{3}{8}$, $15\frac{9}{16}$, $17\frac{3}{4}$, $19\frac{7}{8}$, $21\frac{3}{4}$, $23\frac{3}{4}$,
$25\frac{9}{16}$, $27\frac{3}{8}$, $29\frac{1}{8}$, $30\frac{3}{4}$, $32\frac{1}{2}$, 34,
$35\frac{1}{2}$, $36\frac{7}{8}$, $37\frac{9}{16}$, $38\frac{7}{8}$, $39\frac{7}{8}$, $40\frac{1}{2}$,

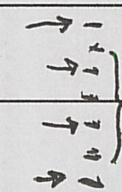
[redacted]

41 3/4, 42 7/8, 43 7/8, 44 13/16, 45 3/4, 46 1/2, 47 1/4,
47 7/8, 48, 48 1/[illegible], 49, 49 1/2, 50, 50 3/8, 50 3/4,

48 1/2

51, 51 5/16, ..., (6 coils)

bottom



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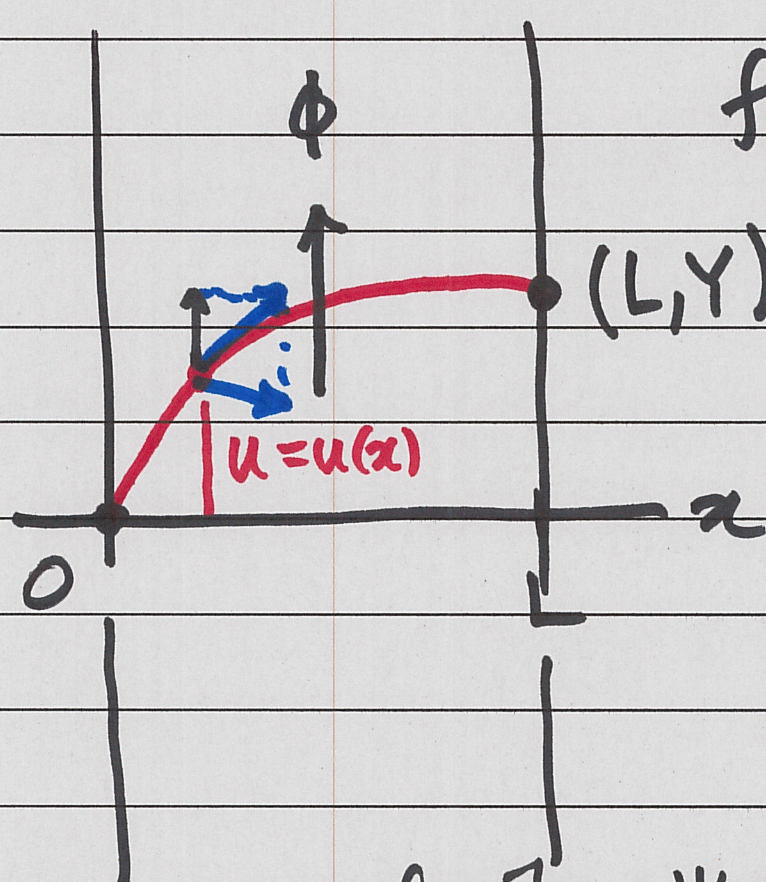
Optimal Navigation (river crossing problem)



flow of river $\phi = \phi(x) > 0$

$\phi(0,1)$

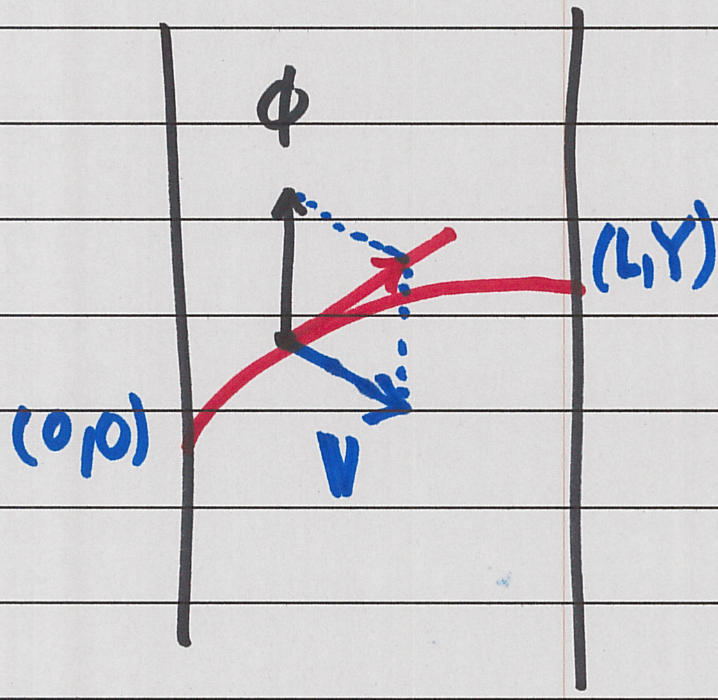
position as a function of time $(x(t), y(t))$

$= (x(t), u(x(t)))$.

BOAT Max velocity $\mathbb{V} = (v_1, v_2)$

$|\mathbb{V}|$ constant V

[redacted]



$$(V_1, V_2 + \phi) = x'(1, u')$$

$$\gamma(t) = (x(t), y(t)) = (x(t), u(x(t)))$$

$$\gamma' = x'(1, u')$$

<u>crossing time</u> : Assume $x' > 0$ so x has an inverse

$$x : [0, T] \to [0, L]$$

$$x^{-1} : [0, L] \to [0, T]$$

crossing time : $T = x^{-1}(L)$.

[redacted]

$$T = x^{-1}(L) = \int_0^L \frac{d}{dx} x^{-1}(\xi)\, d\xi$$

$$\frac{d}{dx} x^{-1} \circ x = \frac{d}{dx} x^{-1} \cdot \underbrace{x'}_{= v_1} = 1$$

$$\frac{d}{dx} x^{-1} = \frac{1}{v_1}$$

$$\boxed{T = \int_0^L \frac{1}{v_1}\, dx} \quad (a).$$

[redacted]