Some comments on the Laplace transform

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The initial value problem (IVP)

$$\left\{\begin{array}{ll} u''+\omega^2 u = f, & t\geq 0 \\ u(0)=0=u'(0) \end{array}\right. \tag{1}$$

is used to model the oscillations of an undamped harmonic oscillator with equilibrium corresponding to $u(0)=0=u'(0)$. It will be noted that the initial conditions stipulate that the system being modeled starts in the equilibrium position, and it is assumed that some motion will be determined by the **forcing function** $f:[0,\infty)\to\mathbb{R}$. The natural space for the forcing function f is perhaps $C^0[0,\infty)$. This is suggested by the assumption that the natural domain for the operator $Lu=u''+\omega^2 u$ is $C^2[0,\infty)$. For many functions $f\in C^0[0,\infty)$ there are various methods to find solutions of (1) including in certain instances using the Laplace transform which is usually defined for a function $f:[0,\infty)\to\mathbb{R}$ which does not grow too rapidly by

$$\mathcal{L}[f]=\int_0^\infty e^{-st}f(t)\,dt. \tag{2}$$

In this formulation $\mathcal{L}[f]$ is thought of as giving the values of a function $F=F(s)$ having various domains according to how the values of the integration work out. Many times one can assume $F:[c,\infty)\to\mathbb{R}$ or $F:\{z\in\mathbb{C}:\text{Re}(z)\geq c\}\to\mathbb{C}$ for some real constant c. The important thing, however, is that from the naive point of view $\mathcal{L}$ transforms a function $f=f(t)$ into a function $F=F(s)$. One of my objectives is to describe a somewhat more unified point of view which is a step in the direction of making rigorous mathematical sense of what is actually happening with the Laplace transform.

In any framework, one crucial property one wishes to have for the Laplace transform $\mathcal{L}$ is that if one knows the value of $\mathcal{L}[u]$ where u is the solution of (1) then one can recover the solution u. In the context of "transform functions" $U=U(s)$ this is accomplished by the **inverse Laplace transform**

$$\mathcal{L}^{-1}[U]=\int e^{st}U(s)$$

where the integral may be some traditional integral with respect to the variable s or something more exotic like a complex line integral. We won't be too interested in the details of this procedure, but we will give examples in which one may be reasonably confident (for various reasons) that there exists a unique function $u:[0,\infty)\to\mathbb{R}$ for which

$$\mathcal{L}[u]=U.$$

A second unifying thread in the story I attempt to present below is the extension of solution techniques to cases in which the forcing has more non-traditional, or less simple, forms. This, on the one hand, motivates the use of the Laplace transform especially in the general framework in which I will cast it. On the other hand, I will begin with situations in which other traditional techniques apply. The simplest example, of course, is when $f\equiv 0$. Please note that in this case, the unique solution of (1) is $u\equiv 0$ modeling the expectant harmonic oscillator which simply persists in equilibrium awaiting in vain the imposition of an inhomogeneity which produces a nontrivial (model) motion.

1 continuous forcing

The problem

$$\begin{cases} u'' + \omega^2 u = \sin \alpha t, & t \geq 0 \\ u(0) = 0 = u'(0) \end{cases} \tag{3}$$

is familiar and interesting to analyze. This problem is often considered broadly representative of models of harmonic oscillators subject to **periodic forcing**. The general solution of the associated homogeneous ODE $u_h'' + \omega^2 u_h = 0$ is

$$u_h(t) = a \cos \omega t + b \sin \omega t$$

where a and b are arbitrary constants. Here we naturally assume $\omega > 0$. A particular solution may intitially be assumed to have the form

$$u_p(t) = c \cos \alpha t + d \sin \alpha t \tag{4}$$

where c and d are some particular constants. This assumption will lead to success in finding a particular solution except in the case when $\alpha = \omega$ in which this assumption leads to failure. Such successes and failures are a familiar part of the technique called "guess and check" or more formally "the method of undetermined coefficients." In this case, substitution of the function given in (4) in the ODE under consideration gives

$$c(\omega^2 - \alpha^2) \cos \alpha t + d(\omega^2 - \alpha^2) \sin \alpha t = \sin \alpha t.$$

One can see immediately why $\alpha = \omega$ is going to lead to failure. But otherwise, we should take $c = 0$ and

$$d = \frac{1}{\omega^2 - \alpha^2}$$

so

$$u_p = \frac{1}{\omega^2 - \alpha^2} \sin \alpha t$$

gives a particular solution. Finally, we can choose a and b in

$$u = u_p + u_h$$

to solve the IVP (3):

$$a = 0$$

$$\omega b + \frac{\alpha}{\omega^2 - \alpha^2} = 0.$$

That is, the unique solution of (3) when $\alpha \neq \omega$ has values

$$u(t) = \frac{1}{\omega^2 - \alpha^2} \left(\sin \alpha t - \frac{\alpha}{\omega} \sin \omega t \right).$$

You can easily check that this function solves (3). There are various forms the expression for the solution u can take to illustrate various oscillatory phenomena, for example "rhythmic beats" among the periodic oscillatons and near resonance. Furthermore, we can formally take the limit as α tends to ω (using L'Hopital's rule) to find

$$\begin{aligned} u(t) &= \lim_{\alpha \to \omega} \frac{1}{-2\alpha} \left(t \cos \alpha t - \frac{1}{\omega} \sin \omega t \right) \\ &= \frac{1}{2\omega} \left(\frac{1}{\omega} \sin \omega - t \cos \omega t \right). \end{aligned}$$

Applying the operator $Lu = u'' + \omega^2 u$ to this function, we find

$$Lu = -\frac{1}{2\omega}(-\omega \sin \omega t - \omega \sin \omega t - \omega^2 t \cos \omega t + \omega^2 t \cos \omega t) = \sin \omega t.$$

Thus, we have found a solution, the fully resonant solution, when $\alpha = \omega$ takes the resonant frequency, and it is easy to check that the solution we have found satisfies the initial conditions as well. This is the unique solution of (3) when $\alpha = \omega$.

Notice there were two somewhat nitpicky cases here: $\alpha \neq \omega$ and $\alpha = \omega$. We now consider solution of the same problem using the Laplace transform.

A small collection of general properties and formulas associated with the naive Laplace transform is required. First of all integrating by parts gives

$$\mathcal{L}[u'] = \int_0^\infty e^{-st} u'(t)\, dt = e^{-st} u(t)\Big|_{t=0}^{\infty} + s \int_0^\infty e^{-st} u(t)\, dt = -u(0) + s\mathcal{L}[u].$$

Applying this technique of integration a second time we get

$$\mathcal{L}[u''] = -u'(0) + s\mathcal{L}[u'] = -u'(0) - su(0) + s^2 \mathcal{L}[u]. \tag{5}$$

Finding $\mathcal{L}[\sin \alpha t]$ is somewhat more complicated, but it is fairly straightforward using the expression/definition

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

One finds

$$\mathcal{L}[\sin \alpha t] = \frac{\alpha}{s^2 + \alpha^2}.$$

In view of the formula (5) one finds a first peculiarity about the application of the Laplace transform. If one naively attempts to "transform" the ordinary differential equation, one is found to actually be rather transforming the entire initial value problem at least in the sense that the initial values are being used. In any case, one finds a presumed relation:

$$s^2 \mathcal{L}[u] + \omega^2 \mathcal{L}[u] = \frac{\alpha}{s^2 + \alpha^2}.$$

Taking this as an algebraic equation for $\mathcal{L}[u]$, which evidently it is, we can write

$$\mathcal{L}[u] = \frac{\alpha}{(s^2 + \alpha^2)(s^2 + \omega^2)}.$$

If $\alpha \neq \omega$, then

$$\begin{aligned} \mathcal{L}[u] &= \frac{\alpha}{\omega^2 - \alpha^2} \left(\frac{1}{s^2 + \alpha^2} - \frac{1}{s^2 + \omega^2} \right) \\ &= \frac{1}{\omega^2 - \alpha^2} \mathcal{L}[\sin \alpha t] - \frac{\alpha}{\omega(\omega^2 - \alpha^2)} \mathcal{L}[\sin \omega t] \\ &= \mathcal{L}\left[\frac{1}{\omega(\omega^2 - \alpha^2)} (\omega \sin \alpha t - \alpha \sin \omega t) \right]. \end{aligned}$$

Thus we arrive at the non-resonant solution. If $\alpha = \omega$ in this case, we proceed differently. We are now starting with

$$\mathcal{L}[u] = \frac{\omega}{(s^2 + \omega^2)^2} = \frac{1}{2s} \frac{2\omega s}{(s^2 + \omega^2)^2} = -\frac{1}{2s} \frac{d}{ds} \left(\frac{\omega}{s^2 + \omega^2} \right).$$

The following two rules are useful at this point:

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}\mathcal{L}[f] \qquad \text{and} \qquad \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] = \mathcal{L}[f]\ \mathcal{L}[g]. \tag{6}$$

We have then

$$\mathcal{L}[u] = \mathcal{L}\left[\frac{1}{2}\right]\ \mathcal{L}[t\ \sin\omega t] = \mathcal{L}\left[\frac{1}{2}\int_0^t \tau\ \sin\omega\tau\,d\tau\right].$$

We conclude

$$\begin{aligned} u(t) &= \frac{1}{2}\left(-\frac{\tau}{\omega}\cos\omega\tau\Big|_{\tau=0}^{t} + \frac{1}{\omega}\int_0^t \cos\omega\tau\,d\tau\right) \\ &= \frac{1}{2}\left(-\frac{t}{\omega}\cos\omega t + \frac{1}{\omega^2}\sin\omega t\right). \end{aligned}$$

This is the resonant solution. Instead of the second rule in (6) we could have used the somewhat simpler rule

$$\mathcal{L}\left[\int_0^t g(\tau)\,d\tau\right] = \frac{1}{s}\mathcal{L}[g].$$

At this point there is little evidence that the method of Laplace transforms for solving initial value problems allows one to do anything one cannot do with more elementary methods. It can perhaps be said that the method allows a more systematic or unified treatment of the cases of non-resonance and resonance considered above. On the other hand, the "method of undetermined coefficients" can be presented in a rather more systematic manner than I have done by including for example a discussion of the null space of the operator $L : C^2[0,\infty) \to C^0[0,\infty)$ with $Lu = u'' + \omega^2 u$, and certainly some level of mystery is involved with the Laplace transform method as well. Specifically, it is not exactly clear what it means to "transform an entire initial value problem." To what set does an initial value problem belong? And in what way is an element of this set being "transformed" into "a function F of a complex variable s?" At least we have illustrated how the Laplace transform method can work in a simple case and produce an outcome we (should) expect.

2 discontinuous forcing

Now that we believe the method of Laplace transforms a little bit and/or see how it works in a simple case, let's try something more exotic.

2.1 delay

Before we do that notice the onset of the continuous forcing considered above may be delayed until some positive time by introducing a discontinuous factor involving a **Heaviside function** $H : \mathbb{R} \to \{0, 1\}$ by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

Given a time of onset t_0, the forcing $f(t) = H(t-t_0)\sin\alpha(t-t_0)$ is still continuous but is delayed in its effect. The Laplace transform of a delayed forcing is given by

$$\mathcal{L}[H(t-t_0)f_0(t-t_0)] = \int_{t_0}^{\infty} e^{-st} f_0(t-t_0)\,dt = \int_0^{\infty} e^{-s(\tau+t_0)} f_0(\tau)\,d\tau = e^{-t_0 s}\mathcal{L}[f_0]. \tag{7}$$

Applying this formula to the initial value problem

$$\begin{cases} u_0'' + \omega^2 u_0 = H(t-t_0)\sin\alpha(t-t_0), & t \geq 0 \\ u_0(0) = 0 = u_0'(0) \end{cases} \tag{8}$$

with delayed forcing simply moves the onset of the forcing to time t_0:

$$\mathcal{L}[u_0] = \frac{\alpha e^{-t_0 s}}{(s^2+\omega^2)(s^2+\omega^2)} = e^{-t_0 s}\mathcal{L}[u]$$

where u is the solution of (3). In this case, we use the inverse Laplace transform in the form

$$\mathcal{L}[u_0] = \mathcal{L}[H(t-t_0)u(t-t_0)]. \tag{9}$$

The result is indicated in Figure 1.

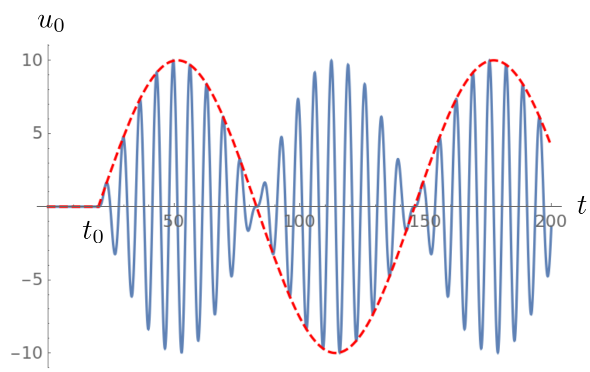

Figure 1: Interacting oscillations from a harmonic oscillator and periodic forcing: rhythmic beats. (Here $\omega = 1$ and $\alpha = 1 - 0.1$. The delay is $t_0 = 20$.)

Exercise 1 Use trigonometric identities to write the functions u and u_0 appearing in (9) in an algebraic form which displays the structure of the rhythmic beats showin in Figure 1. Hint: The frequency of the beats is $(\omega - \alpha)/2$.

Exercise 2 Take the limit as α tends to ω in the solutions u and u_0 appearing in (9) to obtain the fully resonant solution.

Exercise 3 One says that an oscillator modeled by $Lu = u'' + \omega^2 u$ experiences **practical resonance** when the amplitude of oscillations exceeds some specified value M, for example the physical system may cease to function or "break" when oscillations of this amplitude are present. Derive a tolerance condition on α and ω in relation to M giving an analytic condition for modeling practical resonance.

Exercise 4 The forcing in (8) involves a discontinuous function but is still continuous. Verify that the solution u_0 satisfies $u_0 \in C^2[0,\infty)$ and is hence still a classical solution.

2.2 discontinuous forcing

Both methods from the previous section may be applied to the initial value problem

$$\begin{cases} u'' + \omega^2 u = H(t-t_0), & t \geq 0 \\ u(0) = 0 = u'(0). \end{cases} \tag{10}$$

Again, we know the unique (classical) solution for $t \le t_0$ should be $u_0 \equiv 0$. If we assume initial conditions at time t_0 when the Heaviside function becomes nonzero corresponding to a C^1 solution, then we can consider the secondary problem

$$\begin{cases} u'' + \omega^2 u = 1, & t \ge t_0 \\ u(t_0) = 0 = u'(t_0). \end{cases} \tag{11}$$

This problem is easy to solve. A particular solution is given by $u_p(t) = 1/\omega^2$, and we see easily that taking $u_0(t) = [1 - \cos\omega(t - t_0)]/\omega^2$ solves the secondary problem so that

$$u(x) = \begin{cases} 0, & t \le t_0 \\ \dfrac{1 - \cos\omega(t - t_0)}{\omega^2}, & t \ge t_0 \end{cases} \tag{12}$$

is at least a candidate solution for the original problem. See Figure 2. This is not a classical solution

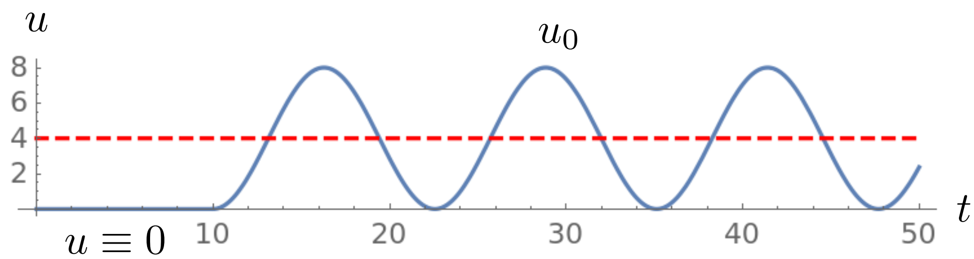

Figure 2: Offset equilibrium in a C^1 weak solution of a harmonic oscillator equation with discontinuous forcing. Here $\omega = 1/2$ and a heaviside function kicks in with amplitude $H(0) = 1$ at time $t_0 = 10$. Notice the oscillations about $u = 1/\omega^2 = 4$ for $t \ge t_0$.

because

$$u''(x) = \begin{cases} 0, & t < t_0 \\ \cos\omega(t - t_0), & t > t_0 \end{cases}$$

with

$$u''(t_0^-) = 0 \qquad \text{and} \qquad u''(t_0^+) = 1 > 0.$$

Thus, this candidate function is not twice differentiable at time $t = t_0$ and cannot be substited into the initial value problem classically. We recall, however, the notion of a **weak C^1 solution** of the ordinary differential equation formulated in this case as follows:

$$\int_{(0,\infty)} (-u'\,\phi' + u\,\phi) = \int_{(0,\infty)} H(t - t_0)\,\phi = \int_{t_0}^{\infty} \phi(t)\,dt \qquad \text{for all } \phi \in C_c^\infty(0, \infty). \tag{13}$$

Exercise 5 Verify that the function u with values given in (12) satisfies

(a) $u \in C^1[0, \infty)$,

(b) $u(0) = 0 = u'(0)$, and

(c) the condition (13) for a weak C^1 solution of the ODE.

Exercise 6 Let $a, x_0, b \in \mathbb{R}$ with $a < x_0 < b$ and functions $p, q, f \in C^0(a, b)$ be given. Show that if $u, w \in C^1(a, b)$ satisfy

(i) $u(x_0) = u_0$, $u'(x_0) = v_0$, and

(ii) for every $\phi \in C_c^\infty(a,b)$

$$\int_a^b [-u'\ \phi' + (p\ u' + q\ u)\phi]\, dx = \int_a^b f\ \phi\, dx,$$

(iii) $w(x_0) = u_0$, $w'(x_0) = v_0$, and

(iv) for every $\phi \in C_c^\infty(a,b)$

$$\int_a^b [-w'\ \phi' + (p\ w' + q\ w)\phi]\, dx = \int_a^b f\ \phi\, dx,$$

then $u \equiv w$. Hint: Show that given any $\psi \in C_c^\infty(a,b)$ and some fixed $\mu \in C_c^\infty(a,b)$ with $\int \mu \neq 0$, there exists a constant c for which $\psi - c\mu = \phi'$ for some $\phi \in C_c^\infty(a,b)$.

Exercise 7 Explain why the assertion of Exercise 6 does not apply to show the uniqueness of the (weak C^1) solution (12) of (10).

Exercise 8 Formulate and prove a uniqueness result that does apply to show the uniqueness of the (weak C^1) solution (12) of (10).

The Laplace transform method, involving integrals rather than derivatives, works seamlessly to produce the unique C^1 weak solution of (10):

$$(s^2 + \omega^2)\mathcal{L}[u] = \mathcal{L}[H(t - t_0)] = \frac{e^{-st_0}}{s}; \qquad \mathcal{L}[u] = \frac{e^{-st_0}}{s(s^2 + \omega^2)}.$$

Therefore,

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{\omega^2} e^{-st_0} \left(\frac{1}{s} - \frac{s}{s^2 + \omega^2} \right) \\ &= \mathcal{L}\left[\frac{1}{\omega^2} H(t - t_0)\,(1 - \cos\omega(t - t_0)) \right].\end{aligned}$$

Here we have used the calculation (7) and/or the rule (9) and the inverse transform related to cosine:

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2 + \omega^2}.$$

Next, let us consider the interesting theoretical[1] problem of subjecting an undamped harmonic oscillator at rest to a constant force of limited duration or a **pulse**. The resulting initial value problem

$$\begin{cases} u'' + \omega^2 u = H(t - t_0) - H(t - t_1), \quad t \geq 0 \\ u(0) = 0 = u'(0) \end{cases} \tag{14}$$

can be found to have a unique C^1 weak solution by direct elementary piecewise analysis, but at this point we go directly to the Laplace transform:

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{\omega^2} e^{-st_0} \left(\frac{1}{s} - \frac{s}{s^2 + \omega^2} \right) - \frac{1}{\omega^2} e^{-st_1} \left(\frac{1}{s} - \frac{s}{s^2 + \omega^2} \right) \\ &= \mathcal{L}\left[\frac{1}{\omega^2} H(t - t_0)\,(1 - \cos\omega(t - t_0)) - \frac{1}{\omega^2} H(t - t_1)\,(1 - \cos\omega(t - t_1)) \right].\end{aligned}$$

[1]We might classify this as a physical problem rather than a theoretical problem, but in principle there seems to always be some form of damping or entropy/loss of energy in actual physical systems. Thus, the very notion of a completely undamped system is somewhat theoretical, though there do seem to be relatively good approximate physical examples involving physical systems in near vacuum or otherwise with a minimum of friction. In any case, by a "theoretical" problem here we do not mean properly a mathematical problem involving equations, but rather a thought experiment about an imagined physical problem to which one may apply mathematical modeling techniques.

We see then that for $t \geq t_1$ we obtain a function

$$u_1(t) = \frac{1}{\omega^2}[\cos\omega(t - t_1) - \cos\omega(t - t_0)]$$

which represents an interaction of wave forms. Naturally, this can be rewritten to display clearly the amplitude and period of the motion for $t \geq t_1$. In fact,

$$\begin{aligned} u_1(t) &= \frac{1}{\omega^2}[(\cos\omega t_1 - \cos\omega t_0)\cos\omega t + (\sin\omega t_1 - \sin\omega t_0)\sin\omega t] \\ &= \frac{\sqrt{2[1 - \cos\omega(t_1 - t_0)]}}{\omega^2}\cos(\omega t - \phi) \end{aligned}$$

where

$$\cos\phi = \frac{\cos\omega t_1 - \cos\omega t_0}{\sqrt{2[1 - \cos\omega(t_1 - t_0)]}} \qquad \text{and} \qquad \sin\phi = \frac{\sin\omega t_1 - \sin\omega t_0}{\sqrt{2[1 + \cos\omega(t_1 - t_0)]}}.$$

A number of interesting properties of the **resultant oscillation**, or the oscillation persisting after the pulse ending at time $t = t_1$, are evident from this expression. The frequency, as one might expect, is the resonant frequency ω of the operator $Lu = u'' + \omega^2 u$ modeling the harmonic oscillator, and also as one might expect, the equilibrium associated with the motion has returned to the natural center $u = 0$. The amplitude

$$\frac{\sqrt{2[1 - \cos\omega(t_1 - t_0)]}}{\omega^2}$$

is itself periodic in t_1 with frequency ω and takes the values between 0 and $2/\omega^2$ inclusive with the largest being exactly twice the amplitude of the forced oscillation given by $u_0(t)$ for $t_0 \leq t \leq t_1$ with respect to the displaced equilibrium; see Figure 2. Thus, such a disturbance can never produce a resultant amplitude of oscillation relative to the central equilibrium $u = 0$ greater than that evident during the period of the pulse. The other extreme

$$\frac{\sqrt{2[1 - \cos\omega(t_1 - t_0)]}}{\omega^2} = 0$$

in resultant magnitude is quite interesting. This means that for pulses ending at discrete times

$$t_1 = t_0 + \frac{2\pi k}{\omega}, \qquad k = 1, 2, 3, \ldots$$

the model predicts that the oscillator returns precisely to the rest state assumed before the onset of the pulse at time $t = t_0$. This of course may happen after any number of nontrivial oscillations during the period of the pulse $t_0 \leq t \leq t_1$.

Exercise 9 Figure 3 illustrates the resultant oscillations associated with unit amplitude square wave pulses with two different durations. Reproduce these illustrations and include them in a series of illustrations with t_1 tending to and including the value $t_1 = 10 + 8\pi$ corresponding to the duration in which the oscillation during the pulse executes exactly two oscillations.

When the duration of the pulse tends to zero of course the solutions themselves also tend uniformly to zero with the resultant amplitude tending to zero in particular. In an attempt to model an **impulsive force** at time t_0, and a persisting resultant non-trivial oscillation for $t > t_0$ in particular, one may consider constant forces of time intervals $t_0 \leq t \leq t_1$ with t_1 tending to t_0 but with **increased amplitude/intensity**. In fact the IVP

$$\begin{cases} u'' + \omega^2 u = a[H(t - t_0) - H(t - t_1)], \quad t \geq 0 \\ u(0) = 0 = u'(0) \end{cases} \tag{15}$$

has solution

$$u_1(t) = \frac{a}{\omega^2}[\cos\omega(t - t_1) - \cos\omega(t - t_0)]$$

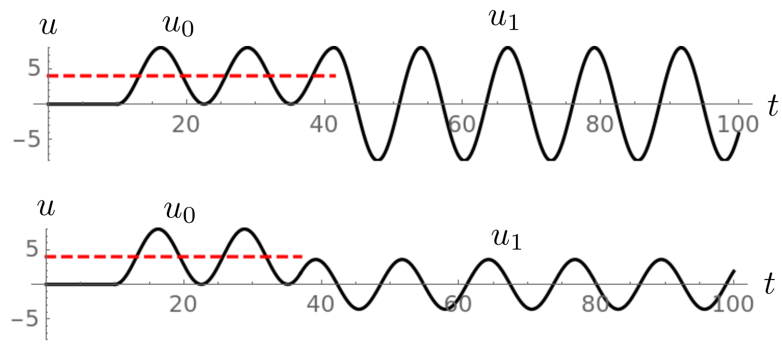

Figure 3: Termination of constant unit forcing in the example of Figure 2 at times $t_1 = 10(1+\pi) \approx 41.4$ (top) and $t_1 = 37$ (bottom). The duration of the pulse corresponding to the interval $t_0 = 10 \le t \le t_1$ is indicated by the red dashed line giving the displaced equilibrium $u = 1/\omega^2$. The resultant amplitude is the maximum possible $2/\omega^2$ in the top plot, and the resultant amplitude is approximately half that value in the bottom plot. In each case the oscillation corresponding to u_0 on the interval $t_0 \le t \le t_1$ of the pulse executes somewhat more than two full oscillations.

and taking $a = 1/(t_1 - t_0)$ we find the limiting resultant oscillation is given by

$$u_*(t) = \lim_{t_1 \searrow t_0} u_1(t) = \frac{1}{\omega} \sin \omega(t - t_0) \neq 0.$$

There are various ways to interpret this function and more specifically

$$u(t) = \frac{H(t-t_0)}{\omega} \sin \omega(t - t_0) = \mathcal{L}^{-1}\left[\frac{e^{-st_0}}{s^2 - \omega^2}\right] \tag{16}$$

as the result of an impulsive force at time $t = t_0$. Let us take first the physical point of view. A plot of this function u is shown in Figure 4. It is most notable that the function $u \in C^0[0,\infty)\backslash C^1[0,\infty)$, so

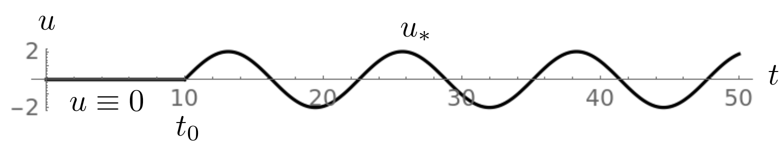

Figure 4: A natural candidate for the model motion associated with an impulsive force.

the concept of a weak C^1 solution cannot be applied here. As we will see below, the elementary solution technique for second order ODEs has no natural application in this context either. In particular,

$$u'(t_0^-) = 0 \qquad \text{and} \qquad u'(t_0^+) = 1 > 0. \tag{17}$$

The resultant amplitude $1/\omega$ is also interesting.

Exercise 10 Make a series of plots of the function

$$u(t) = \frac{1}{(t_1 - t_0)\omega^2}\left[H(t-t_0)\,(1 - \cos \omega(t-t_0)) - H(t-t_1)\,(1 - \cos \omega(t - t_1))\right]$$

for values t_1 tending to t_0. Explain what initial value problem this function solves and in what sense. Can you see a relation between the amplitude(s) associated with this funtion u and the amplitude of the limiting function u_*?

From the physical point of view the function u in (16) models/suggests a jump in the velocity of magnitude 1 given explicitly in (17). This is most often interpreted as a jump in **momentum** of magnitude $1 = m\ u'(t_0^+)^2$. Thus, one is said to be dealing with a **unit impulse** in this limit responsible for a concentrated change in momentum of one unit. More generally, one may speak (physically) about an **impulse at time** $t = t_0$ **responsible for a discontinuous change in momentum of any particular value** p_0. One may then work out the corresponding change in velocity and formulate a secondary initial value problem.

Exercise 11 Consider the operator $L : C^2[0, \infty) \to C^0[0, \infty)$ with $Lu = mu + \omega^2 u$ for some $m, \omega > 0$. Find an appropriate value v_0 to complete the secondary IVP

$$\begin{cases} mu_*'' + \omega^2 u_* = 0, \quad t \geq t_0 \\ u_*(t_0) = 0 \\ u_*'(t_0) = v_0 \end{cases} \tag{18}$$

so that the solution $u(t) = H(t - t_0)\ u_*(t)$ models a jump in momentum by a specified value p_0.

It is quite natural, though technically incorrect, to write down an initial value problem of the form

$$\begin{cases} u' + \omega^2 u = \delta(t - t_0), \quad t \geq t_0 \\ u(0) = 0 = u'(0) \end{cases} \tag{19}$$

which is intended to have the limiting function with values given in (16) as a solution. Here, the inhomogeneity $\delta(t - t_0)$ involves what is called a "Dirac delta function," having the property that

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0 \qquad \text{but} \qquad \int_0^\infty f(t)\, \delta(t - t_0)\, dt = f(t_0) \tag{20}$$

at least for every function $f \in C^0[0, \infty)$. This is objectionable mathematically for various reasons. First of all any function satisfying the first condition in (20) has

$$\int \delta = \int f\delta = 0.$$

As noted above, one can generalize the notion of solutions of $Lu = u'' + \omega^2 u = f$ to certain cases in which $u \in C^1[0, \infty)$. One could also attempt to formulate a weak C^0 solution for a function $u \in C^0[0, \infty)$. This might start with the condition

$$\int_0^\infty (u\phi'' + \omega^2 u\phi)\, dt = \int_0^\infty f\phi\, dt \qquad \text{for every } \phi \in C_c^\infty[0, \infty).$$

Two problems remain. The first is making sense of the initial condition $u'(0) = 0$ for such a weak solution, which may not have a classical first derivative. The second more serious problem relates back to the fundamental error of assuming there is a function with the properties (20). What one really wishes to have is

$$\int_0^\infty (u\phi'' + \omega^2 u\phi)\, dt = \phi(t_0) \qquad \text{for every } \phi \in C_c^\infty[0, \infty). \tag{21}$$

There is no function f for which (21) holds.

Exercise 12 Show that if $f \in L^1_{loc}[0, \infty)$ and condition (21) holds, then $f(t) = 0$ for $t \neq 0$, and hence

$$\int_0^\infty f\phi\, dt = 0 \qquad \text{for every } \phi \in C_c^\infty[0, \infty),$$

and so arrive at a contradiction.

Nevertheless, condition (21) is relatively close to a sensible formulation of the problem. What is needed, however, is not really a notion of a weak solution, but rather that of a distributional solution.

We will consider another problem with the Dirac delta *function,* in the next section.

In some sense, whenever (19) is written formally, the only sensible interpretation is the classical introduction of a jump discontinuity in the velocity, or in the momentum respectively, as represented by the auxiliary problem (18).

3 impulsive forcing and distributions

If we "reverse engineer" formula (16) as it is usually applied in solving the incorrect initial value problem (19), then we conclude

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}. \tag{22}$$

This is a formula one finds in most any table of Laplace transforms and is probably familiar to you. It is also incorrect. If the δ *function* is bounded by some number $M > 0$ so that

$$|\delta(t)| < M \qquad \text{for} \qquad 0 < t < \infty,$$

which surely must be the case according to the first property in (20), then

$$\mathcal{L}[\delta] = \int_{(0,\infty)} \delta(t)e^{-st}\,dt \leq M \int_{(0,\infty)} e^{-st}\,dt = \frac{M}{s}.$$

This means in particular that

$$\lim_{s\nearrow\infty} \mathcal{L}[\delta](s) = 0.$$

On the other hand, evaluation of (22) at $t_0 = 0$ gives

$$\mathcal{L}[\delta](s) \equiv 1.$$

Thus, we have yet another nominal contradiction. In view of all these nominal contradictions, one may ask: Can a mathematical meaning be attached to (19)? The answer to this question provided by Laurent Schwartz is that, first of all, one needs to move outside the context of real valued functions of a real variable. Fortunately, There is a context or "world" in which all, or at least most of, these contradictions go away. This "world" is a somewhat more abstract world, but fundamentally any weak formulation like (21) though a step in the right direction is not a statement about (or involving) just real valued functions of a real variable. In short, there is no such thing as a Dirac delta "function" in the sense of (20).

Consider the following interesting observation about functions:

Associated with each function $f \in C^0(0,\infty)$ there exists an **integral functional** $\mathcal{F} : C_c^\infty(0,\infty) \to \mathbb{R}$ given by

$$\mathcal{F}[\phi] = \int_0^\infty f(t)\phi(t)\,dt.$$

More importantly for our purposes (or for the sake of inversion of something like the Laplace transform) knowing the values of the functional $\mathcal{F} : C_c^\infty(0,\infty) \to \mathbb{R}$ determines the function f. This is a manifestation of the fundamental lemma of vanishing integrals, which is also called the fundamental lemma of the calculus of variations, and might be called the fundamental lemma of integral functionals.

Let $\mathcal{D}$ be the space of **continuous linear functionals** $\mathcal{G} : C_c^\infty(0,\infty) \to \mathbb{R}$. Note carefully the fundamental assertion:

Theorem 1 If $\mathcal{F} \in \mathcal{D}$ and **it is known** that there exists some continuous function $f \in C^0(0,\infty)$ such that

$$\mathcal{F}[\phi] = \int_0^\infty f(t)\phi(t)\,dt \qquad \text{for every } \phi \in C_c^\infty(0,\infty),$$

then the function f is uniquely determined.

Exercise 13 Prove Theorem 1. Hint: If $f_1, f_2 \in C_c^\infty(0,\infty)$ and

$$\mathcal{F}[\phi] = \int_0^\infty f_j(t)\phi(t)\,dt \qquad \text{for } j = 1, 2 \text{ and every } \phi \in C_c^\infty(0,\infty),$$

then

$$\int_0^\infty [f_1(t) - f_2(t)]\phi(t)\,dt = 0 \qquad \text{for every } \phi \in C_c^\infty(0,\infty),$$

The next crucial point is that there are elements of the space (world) of distributions that are not given by integral functionals. One of those is the **evaluation functional** $\mathcal{E}_{t_0} : C_c^\infty(0,\infty) \to \mathbb{R}$ given by

$$\mathcal{E}_{t_0}[\phi] = \phi(t_0).$$

This, as far as I know, is basically the only "real," i.e., only sensible, Dirac delta function.

3.1 some technical details

I've suggested two of the big ideas, but I don't want to go too much further before at least mentioning a thing or two I've swept under the rug. The first is that you probably have no idea what it means for a functional $\mathcal{F} : C_c^\infty(0,\infty) \to \mathbb{R}$ to be **continuous**. Linearity is simple enough because $C_c^\infty(0,\infty)$ is a vector space, but to make sense of continuity one needs something like a notion of distance between functions in $C_c^\infty(0,\infty)$. I am sorry to report that there is no reasonable, or at least "nice," notion of distance between functions in $C_c^\infty(0,\infty)$. At least I do now know of one. Perhaps you can find one. Fortunately, there is also a broader context in which continuity makes sense. You may recall that we have mentioned inner product spaces and normed vector spaces and metric distance spaces, and the fact that each induces the structure of the next in the sense that every inner product space has a norm and every normed space has a metric distance. There is a fourth kind of space called a **topological space** in which one need not have a metric distance but one still has notions of open sets, continuity, and convergence. Every metric space is a topological space, and $C_c^\infty(0,\infty)$ can be made into a topological space or endowed with a topology, so to speak. In fact, $C_c^\infty(0,\infty)$ can be made a normed space because it is a subspace of $C^k(0,\infty)$ for each $k = 1, 2, 3, \ldots$ with the norm

$$\|\phi\|_{C^k} = \sum_{j=0}^k \left\| \frac{d^j\phi}{dt^j} \right\|_{L^\infty(0,\infty)} = \sum_{j=0}^k \sup_{t\in(0,\infty)} \left| \frac{d^j\phi}{dt^j} \right|.$$

Unfortunately, none of these norms gives a topology in which convergence implies the convergence of all derivatives. As you might imagine, however, there does exist a topology with respect to which convergence of a sequence $\{\phi_j\}_{j=1}^\infty$ of a sequence of functions in $C_c^\infty(0,\infty)$ to a function $\phi \in C_c^\infty(0,\infty)$ does imply (some kind of) convergence of the derivatives of all orders.

From the point of view of linear spaces the space of distributions is the **dual space**, technically the *continuous dual space,* associated with the vector space $C_c^\infty(0,\infty)$. The point is that to make sense of the continuous dual space, one needs a topology on the collection of test functions $C_c^\infty(0,\infty)$. The question of defining the topology, i.e., open sets, is also made somewhat complicated because the compact support of the functions involved also plays a role, i.e., it is not just the convergence of all derivatives in a sequence one is after but rather the convergence of all derivatives on compact subsets (or something like that). The order and how you phrase these requirements in terms of open sets of functions must be done rather carefully.

This aspect of the theory of functionals/distributions is somewhat difficult, and I will not provide further detail here, except to say that there are books on **topological spaces** without further structure and advanced texts in the subject of functional analysis usually contain a treatment of the relevant spaces called **topological vector spaces**; there are also entire texts just on topological vector spaces. One can read such books.

Once one makes sense of what it means for a functional in a space like $\mathcal{D}$ to be continuous, then one can introduce a linear structure on $\mathcal{D}$ itself and make the space of distributions into a topological vector space. Once this is done one can show a sequence of integral functionals (corresponding to unit square pulses) with

$$\mathcal{F}_j[\phi] = 2j \int_{(0,\infty)} \chi_{[t_0-1/j, t_0+1/j]}\ \phi = 2j \int_{t_0-1/j}^{t_0+1/j} \phi(t)\, dt$$

converges to the evaluation functional $\mathcal{E}_{t_0}$. The family of standard mollifiers $\{\mu_\sigma(t-t_0)\}_{\sigma>0}$ corresponds to a family of integral functionals with the same limit. Again, note carefully what is happening here. A sequence of functionals lying in the subspace of functionals corresponding to actual real valued functions of a real variable converge in the space of functionals to the evaluation functional, which is a more exotic object lying in the space of distributions $\mathcal{D}$ but outside the subspace corresponding to functions.

A second technical point is that while Theorem 1 is stated for application to continuous functions, the much larger space to which the basic assertion applies, and the more natural space to which it applies, is the space of locally integrable functions.

Theorem 2 If $\mathcal{F} \in \mathcal{D}$ and **it is known** that there exists some function $f \in L^\infty_{loc}(0,\infty)$ such that

$$\mathcal{F}[\phi] = \int_0^\infty f(t)\phi(t)\, dt \qquad \text{for every } \phi \in C_c^\infty(0,\infty),$$

then the function f is uniquely determined.

Thus, the world of functions (or if you like functionals corresponding to functions) is not properly the world of continuous functions but is rather much larger. There are of course some real valued functions with domain $(0,\infty)$ which are not included in this discussion. There are, for example, the non-measurable functions. One hardly ever sees these. There are also some functions with large values "everywhere" so that they do not admit reasonable integration techniqes to be applied to them. Such unruly functions will have to be left out of the discussion, but almost any function $f : (0,\infty) \to \mathbb{R}$ I can think of will be found in $L^1_{loc}(0,\infty)$.

Finally, it may be considered a technical detail that there are other, usually much larger, collections of test functions which may be considered. I will mention two of them: There is the **Schwartz class**

$$\mathcal{S} = \left\{ \phi \in C^\infty(0,\infty) : \sup_{n,j\in\mathbb{N}, t\in(0,\infty)} \left| t^n \frac{d^j\phi}{dt^j} \right| < \infty \right\}.$$

The condition defining the Schwartz class is that the function and all its derivatives decay faster than any power of t.

Taking the dual space of $\mathcal{S}$ one obtains what are called the **tempered distributions** $\mathcal{S}^*$, or Schwartzian distributions, which are a subspace of $\mathcal{D} = (C_c^\infty(0,\infty))^*$. These are of particular significance if one wants to make mathematical sense of the **Fourier transform**. It is often the case that the C_c^∞ functions can approximate other test functions. Note in particular that $C_c^\infty(0,\infty) \subset \mathcal{S}$ and $C_c^\infty(0,\infty)$ is "dense" as a topological subspace in the sense that the closure of $C_c^\infty(0,\infty)$ in $\mathcal{S}$ is all of $\mathcal{S}$.

The other collection of test functions I'll mention is

$$\mathcal{W} = \{e^{-st} : s > 0\}.$$

That is technically we are considering the functions $\exp_s : (0,\infty) \to \mathbb{R}$ with values

$$\exp_s(t) = e^{-st}.$$

These "negative exponential" test functions make an appearance in the naive Laplace transform. Note that casting

$$F(s) = \int_{t\in(0,\infty)} e^{-st} f(t) \tag{23}$$

into the framework of test functions and linear functionals, the "function of s" with values given by $F(s)$ is simply giving the functional values

$$\mathcal{F}[\phi] = \int_{(0,\infty)} f\phi \tag{24}$$

for the particular test functions $\phi \in \mathcal{W}$. The collection of functions $\mathcal{W}$ is not a vector space.

Exercise 14 Plot (some of) the test functions in the collection $\mathcal{W}$.

(a) If you know the values of the functional associated with $f \in C^0(0,\infty)$ in (24) for every $\phi \in C_c^\infty(0,\infty)$, then do you know the values $F(s)$ given in (23)?

(b) If $f \in C^0(0,\infty)$ and you know the values $F(s)$ given in (23) for every $s \in (0,\infty)$, then do you know $\mathcal{F}[\phi]$ given in (24) for every $\phi \in C_c^\infty(0,\infty)$, i.e., do you know the function f?

Hint: What happens if you mollify $\exp_s$?

3.2 distributional ODEs

We have then an exotic space $\mathcal{D}$ containing a subspace corresponding to the functions $f : (0,\infty) \to \mathbb{R}$ with which we are familiar and containing also some more exotic creatures like the evaluation functional. I would like to finish by giving a rough idea of how one can translate an operator like $Lu = u'' + \omega^2 u$, an ODE like $Lu = f$, and an initial value problem like

$$\begin{cases} u'' + \omega^2 u = f(t), & t > 0 \\ u(0) = u'(0) = 0 \end{cases} \tag{25}$$

into the world of distributions.

A first (amazing) thing to note is that derivatives make sense for distributions. Say we start with a differentiable function $f \in C^1(0,\infty)$. Then there is certainly a corresponding functional $\mathcal{F} \in \mathcal{D}$ with

$$\mathcal{F}[\phi] = \int_{(0,\infty)} f\phi.$$

If there is a derivative for $\mathcal{F}$ in $\mathcal{D}$, then it should be the functional $\mathcal{F}' : C_c^\infty(0,\infty) \to \mathbb{R}$ with

$$\mathcal{F}'[\phi] = \int_{(0,\infty)} f'\phi.$$

Notice that we can integrate by parts in this case to find

$$\mathcal{F}'[\phi] = -\int_{(0,\infty)} f\phi' = -\mathcal{F}[\phi'].$$

This remarkable relation can be applied to **absolutely any** distribution: Given a continous linear functional $\mathcal{G} : C_c^\infty(0,\infty) \to \mathbb{R}$, the evaluation functional for example, we define the derivative of $\mathcal{G}$ to be the functional $\mathcal{G}' \in \mathcal{D}$ given by

$$\mathcal{G}'[\phi] = -\mathcal{G}[\phi'].$$

The weak formulation of the ODE $Lu = u'' + \omega^2 u = f$ is now easy: Find a linear functional $\Upsilon \in \mathcal{D}$ for which

$$\Upsilon'' + \omega^2 \Upsilon = \mathcal{F} \tag{26}$$

where $\mathcal{F}$ is the integral functional in $\mathcal{D}$ associated with the function f. On the face of it, solutions of the equation in (26) could lie anywhere in the wide world of distributions $\mathcal{D}$. If $f \in C^0[0,\infty)$, then we know lots of solutions corresponding to integral functionals (and about their structure as a set) by the existence and uniqueness theorem for linear ODEs. If f is discontinuous, but still an actual function $f : (0,\infty) \to \mathbb{R}$, then we have various weak formulations that are included in (26).

Exercise 15 Show a weak C^1 solution u of the initial value problem (25) corresponds to an integral functional

$$\Upsilon[\phi] = \int_{(0,\infty)} u\phi$$

which solves (26).

But most importantly there are distributional ODEs which do not correspond to any ordinary differential equation for (or involving) functions. An example of such a distrubutional ODE is

$$\Upsilon'' + \omega^2 \Upsilon = \mathcal{E}_{t_0} \tag{27}$$

where $\mathcal{E}_{t_0}$ is the evaluation functional.

Exercise 16 Show that if $u \in C^0(0,\infty)$ corresponds to an integral functional

$$\Upsilon[\phi] = \int_{(0,\infty)} u\phi$$

then Υ solves (27) if and only if the condition (21) holds.

Exercise 17 Show the function $u \in C^0(0,\infty)$ with values given in (16) corresponds to an integral functional $\Upsilon \in \mathcal{D}$ with

$$\Upsilon[\phi] = \int_{(0,\infty)} u\phi$$

which solves (27).

If we pair the distributional ODE $\Upsilon'' + \omega^2 \Upsilon = \mathcal{E}_{t_0}$ with the initial condition $u(0) = 0$ for a continuous function $u \in C^0(0,\infty)$, then we are rather close to a formulation of (19) for a "C^0 distributional solution" which makes sense. There is one final detail which is the consideration of the initial condition $u'(0) = 0$. I believe the correct way to do this is to consider integral functionals Υ corresponding to functions u in the closure of $C_c^\infty(0,\infty)$ in $C_c^0(0,\infty)$. This involves consideration of the topology (or some topology) on $C_c^0(0,\infty)$ which is even more complicated than the topology on $C_c^\infty(0,\infty)$. Note: One definitely does not want the C^0 norm topology on $C_c^0(0,\infty)$ because $C_c^\infty(0,\infty)$ is not dense in that topology. For example, the function given in (16) cannot be approximated in the C^0 norm by functions in $C_c^\infty(0,\infty)$.

In any case, I will leave this last difficulty and the relation between the appearance of initial values $u(0)$ and $u'(0)$ in the naive Laplace transform formula

$$\mathcal{L}[u''] = -u(0) - su'(0) + s^2 \mathcal{L}[u]$$

and the "real" Laplace transform $\mathcal{L} : L^1_{loc}(0,\infty) \to \mathcal{D}$ to your contemplation.

Exercise 18 Consider the continuous function $u \in C^0[0,\infty)$ with values given in (16) and the associated integral functional $\Upsilon \in \mathcal{D}$ with values

$$\Upsilon[\phi] = \int_{(0,\infty)} u\phi.$$

What is the relation between Υ'' and the collection of values

$$U''(s) = -u(0) - su'(0) + s^2 U(s)$$

where

$$U(s) = \int_0^\infty e^{-st} u(t)\, dt \qquad ?$$

In particular, can $u'(0)$ be determined from Υ for u in some appropriate class of functions $\mathcal{C}$ with

$$C^1(0,\infty) \subsetneqq \mathcal{C} \subsetneqq C^0(0,\infty) \qquad ?$$

Formulate a notion of "$\mathcal{C}$ weak distributional solutions" of the distributional IVP

$$\begin{cases} \Upsilon'' + \omega^2 \Upsilon = \mathcal{G}, & t > 0 \\ u(0) = u'(0) = 0 \end{cases} \tag{28}$$

where $\mathcal{G}$ is a general element of the space of distributions $\mathcal{D}$.