The Volume of Unit Balls

John McCuan

April 23, 2023

These are some notes related to Problem 1 of Assignment 14 from MATH 6702 in Spring semester 2023. I know I've assigned this problem in several other semesters, and I'm sure I've typed up a solution for it before. Hopefully, this version will be easier to find in the future. I'm going to start with a more specific question posed this semester by Sabrina Schneider. Here is Sabrina's question:

What is the 4-dimensional measure of the unit ball in $\mathbb{R}^4$?

I plan to derive a satisfying answer is several (slightly) different ways. Before beginning, let me remind you that we have a notation for such quantities: We let the symbol ω_n denote the n-dimensional measure of the n-dimensional unit ball

$$B_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\} \subset \mathbb{R}^n,$$

so that

$$\omega_n = \mathcal{L}^n(B_1) = \int_{B_1} 1.$$

I've written B_1 for $B_1(\mathbf{0})$ here, but the center does not really matter. The measure $\mathcal{L}^n$ is called n-dimensional Lebesgue measure. The radius does matter, and you should be able to change variables in an n-dimensional integral to show

$$\mathcal{L}^n(B_r) = \omega_n \, r^n.$$

You may know $\omega_2 = \pi$, $\omega_3 = 4\pi/3$, and even $\omega_1 = 2$, but we'll verify these also below.

1 First Answer

Let's start with $n = 3$ and review spherical coordinates. The spherical coordinates map $\Psi : [0, \infty) \times [0, \pi] \times \mathbb{R} \to \mathbb{R}^3$ is given by

$$\Psi(r, \phi, \theta) = r(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi).$$

The angle ϕ is called the **azimuthal angle** and is measured from the positive x_3 or z-axis. The parameters r and θ are the familiar **radius** and **polar angle**. In this case, the polar angle is measured between a half-plane containing the x_3-axis and the $x_1 > 0 = x_2$ half-plane. Perhaps you can/should draw a picture of a typical point $\mathbf{x}$ in the first octant of $\mathbb{R}^3$ and the spherical parameters r, ϕ, and θ associated with the point $\mathbf{x}$.

Using this map as a parameterization of the unit ball in $\mathbb{R}^3$ on $[0, 1) \times [0, \pi) \times [0, 2\pi)$, we note

$$D\Psi = \begin{pmatrix} \sin \phi \cos \theta & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \cos \phi \sin \theta & r \sin \phi \cos \theta \\ \cos \phi & -r \sin \phi & 0 \end{pmatrix},$$

$$\det(D\Psi) = \cos \phi \det \begin{pmatrix} r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ r \cos \phi \sin \theta & r \sin \phi \cos \theta \end{pmatrix}$$

$$+ r \sin \phi \det \begin{pmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta \end{pmatrix}$$

$$= r^2 \sin \phi,$$

and we can write the volume of the 3-ball as an iterated integral:

$$\omega_3 = \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin \phi \, d\phi \right) \left(\int_0^1 r^2 \, dr \right)$$

$$= \frac{2\pi}{3} (- \cos \phi) \Big|_{\phi=0}^{\pi}$$

$$= \frac{4\pi}{3}.$$

If you do not follow all details of the next calculation, do not worry, the basic procedure will be repeated several times below. For the unit ball in $\mathbb{R}^4$ we introduce

a **secondary azimuthal angle** ϕ_2 with $0 \leq \phi_2 < \pi$. Each value of ϕ_2 determines a "height" $x_4 = \cos\phi_2$ in $\mathbb{R}^4$. (Think about what happens with the azimuthal angle ϕ in the 3-dimensional case.) In this way, we can parameterize a three-dimensional cross-section of

$$B_1(\mathbf{0}) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 < 1\}$$

given by
$$C(\phi_2) = \{(x_1, x_2, x_3, \cos\phi_2) : x_1^2 + x_2^2 + x_3^2 < 1 - \cos^2\phi_2\}.$$

It is rather convenient that $C(\phi_2)$ is a 3-sphere of radius $\sin\phi$. In particular, we can introduce modified spherical coordinates to parameterize $B_1(\mathbf{0}) \subset \mathbb{R}^4$ as

$$X(r, \phi, \theta, \phi_2) = ((r\sin\phi_2)\sin\phi\cos\theta, (r\sin\phi_2)\sin\phi\sin\theta, (r\sin\phi_2)\cos\phi, \cos\phi_2)$$

where $(r, \phi, \theta, \phi_2) \in [0, 1) \times [0, \pi) \times [0, 2\pi) \times [0, \pi)$. Notice that the first three components parameterize $C(\phi_2)$ and both this parameterization as well as the entire parameterization are non-singular except for a set of measure zero.[1] By our general principles of integration and scaling factors, we (should) know we can compute ω_4 as an iterated integral using this parameterization. In order to compute the scaling factor associated with this map, we observe

$$DX = \begin{pmatrix} \sin\phi_2(D\Psi)_{3\times3} & \mathbf{0}_{3\times1} \\ \\ \mathbf{0}_{1\times3} & -\sin\phi_2 \end{pmatrix}.$$

Expanding along the last row and noting that $\det[\sin\phi_2 D\Psi] = \sin^3\phi_2 \det(D\Psi)$, we find
$$\det(DX) = \sin^4\phi_2 \det(D\Psi) = r^2 \sin^4\phi_2 \sin\phi.$$

Therefore,

$$\omega_4 = \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^1 r^2 \sin^4\phi_2 \sin\phi \, dr \, d\phi \, d\theta \, d\phi_2 = \frac{4\pi}{3} \int_0^\pi \sin^4\phi_2 \, d\phi_2.$$

The integral appearing here falls into a general class of integrals we should consider later, but for now, let's just do this one. I am going to use a different variable name

---

[1]This will become even more apparent once we compute the scaling factor.

writing $\phi_2 = \theta$ just to avoid writing the subscript; I do not mean by this that θ is a polar angle.

$$\int_0^\pi \sin^4 \theta \, d\theta = \int_0^1 \left(\frac{1 - \cos(2\theta)}{2} \right)^2 d\theta$$
$$= \frac{1}{4} \int_0^\pi [1 - 2\cos(2\theta) + \cos^2(2\theta)] \, d\theta$$
$$= \frac{\pi}{4} - \sin(2\theta)\Big|_0^\pi + \frac{\pi}{8}.$$

The last integral here has value half of what you would get if the integrand were the constant function $f \equiv 1$ because $\cos^2(2\theta) + \sin^2(2\theta) \equiv 1$ and

$$\int_0^\pi \sin^2(2\theta) \, d\theta = \int_0^\pi \cos^2(2\theta) \, d\theta$$

as you should check using trigonometric identities/symmetry. The middle evaluation is of course zero because sine is 2π periodic. Thus,

$$\int_0^\pi \sin^4 \theta \, d\theta = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$$

and

$$\omega_4 = \frac{4\pi}{3} \frac{3\pi}{8} = \frac{\pi^2}{2}.$$

Neat huh?

2   More

To show the versatility of our techniques of integration (and maybe even what might be called "power") I'm going to rederive ω_4 using a slightly different parameterization. I'll also start with the next lower dimension and apply the basic approach starting from there.

Take, for example, the disk $B_1(\mathbf{0}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \subset \mathbb{R}^2$. Rather than the usual polar angle/polar coordinates in the disk, I'm going to introduce an azimuthal angle. In fact, the approach I'm about to discuss is sometimes called the method of general polar coordinates, but a better name would be the **method of azimuthal coordinates**. Let θ_2 be the angle made by a ray from the origin with the

positive x_2-axis (that is, the y-axis). We'll use the usual sign convention so that the first quadrant and the right half plane correspond to $0 \le \theta_2 \le \pi/2$ and $0 \le \theta_2 \le \pi$ respectively. (Draw pictures!) Each value of θ with $0 < \theta_2 < \pi$ determines a height $x_2 = \cos \theta_2$, and the entire (open) disk may be decomposed into "balls"

$$C(\theta_2) = \{(x, \cos \theta_2) : -\sin \theta_2 < x < \sin \theta_2\}$$

of one lower dimension. You may note (and you should note) that balls in one lower dimension happen to be intervals. This decomposition of the disk into intervals, as opposed to the polar decomposition into rays and circles, suggests the (azimuthal) parameterization $X : (-1, 1) \times (0, \pi) \to \mathbb{R}^2$ given by

$$X(t, \theta_2) = (t \sin \theta_2, \cos \theta_2).$$

You may note (and you should note) that $(-1, 1)$ is the unit ball in $\mathbb{R}^1$. We conclude

$$\omega_2 = \int_0^\pi \int_{-1}^1 |\det(DX)| \, dt \, d\theta_2 = \pi.$$

Remember the scaling factor of a full-dimension non-singular parameterization is given by the **absolute value** of the determinant of the full-derivative of the parameterization, and in this case

$$DX = \begin{pmatrix} \sin \theta_2 & t \cos \theta_2 \\ 0 & -\sin \theta_2 \end{pmatrix} \qquad \text{with} \qquad \det(DX) = -\sin^2 \theta_2.$$

Notice we can also write

$$\omega_2 = \int_0^\pi \left(\int_{C_1} \sin^2 \theta_2 \right) d\theta_2 = \int_0^\pi \left(\sin \theta_2 \int_{C_1} 1 \right) \sin \theta_2 \, d\theta_2$$

where $C_1 = B_1(0) \subset \mathbb{R}^1$ is the unit ball in $\mathbb{R}^1$, one factor of $\sin \theta_2$ comes from scaling of the unit interval C_1 to be the same size as the intervals in the decomposition of the disk, and the other factor of $\sin \theta_2$ arises directly as a scaling from the azimuthal parameter θ_2 in the disk. This is a more sophisticated way to express our application of Fubini's theorem here, and it will be used (freely and extensively) in the next section.

In three dimensions, we have here an alternative to spherical coordinates with two azimuthal angles. The angle θ_3 is the standard azimuthal angle in $\mathbb{R}^3$. Each θ_3 with $0 < \theta_3 < \pi$ determines a horizontal disk

$$C(\theta_3) = \{\sin \theta_3 (c_1, c_2, 0) + \cos \theta_3 : c_1^2 + c_2^2 < 1\}.$$

5

Thus we may consider $X : C_2 \times (0, \pi) \to \mathbb{R}^3$ by

$$X(\mathbf{p}, \theta_3) = ((\sin \theta_3) \, \mathbf{p}, \cos \theta_3)$$

where $C_2 = B_1(\mathbf{0}) \subset \mathbb{R}^2$ is the unit disk in $\mathbb{R}^2$. Less abstractly, we can incorporate the azimuthal parameterization of C_2 and write $X : (-1, 1) \times (0, \pi) \times (0, \pi) \to \mathbb{R}^3$ by

$$X(t, \theta_2, \theta_1) = ((\sin \theta_3)t \sin \theta_2, (\sin \theta_3) \cos \theta_2, \cos \theta_3).$$

It follows that

$$\omega_3 = \int_0^\pi \left(\sin^2 \theta_3 \int_{C_2} 1 \right) \sin \theta_3 \, d\theta_3 \qquad (1)$$

$$= \pi \int_0^\pi \sin^3 \theta_3 \, d\theta_3.$$

Since

$$\int_0^\pi \sin^3 \theta \, d\theta = \int_0^\pi (1 - \cos^2 \theta) \sin \theta \, d\theta = 2 - \frac{2}{3} = \frac{4}{3},$$

we conclude

$$\omega_3 = \pi \, \frac{4}{3} = \frac{4\pi}{3}$$

as obtained above. Notice the scaling factors in (1). There is a quadratic factor $\sin^2 \theta_3$ for integration on the unit disk because the radius of the horizontal slice of $B_1(\mathbf{0}) \subset \mathbb{R}^3$ is a disk of radius $\sin \theta_3$. There is also a scaling by $\sin \theta_3$ in the direction of $\mathbf{e}_3$ directly due to the use of the third azimuthal angle, i.e., the usual azimuthal parameterization in $\mathbb{R}^3$.

Now we should see a pattern: $X : C_3 \times (0, \pi) \sim (-1, 1) \times (0, \pi)^3 \to B_1(\mathbf{0}) \subset \mathbb{R}^4$ by

$$X(\mathbf{p}, \theta_4) = \sin \theta_4 \, (\mathbf{p}, 0) + \cos \theta_4 \, \mathbf{e}_4,$$

that is,

$$X(t, \theta_2, \theta_3, \theta_4) = ((\sin \theta_4) \, \mathbf{p}, \cos \theta_4),$$

so that

$$\omega_4 = \int_0^\pi \left(\sin^3 \theta_4 \int_{C_3} 1 \right) \sin \theta_4 \, d\theta_4 = \frac{4\pi}{3} \int_0^\pi \sin^4 \theta_4 \, d\theta_4.$$

We made this calculation in the last section:

$$\omega_4 = \frac{\pi^2}{2}.$$

3   Even more

In the homework assignment I have given you a general unified formula for ω_n in terms of the gamma function. And it is usual to do this. I prefer, however, some alternative formulas that make a distinction between the even and odd dimensions. we have the base cases above, and we have a general inductive procedure based on azimuthal angle:

$$\omega_n = \int_0^\pi \left(\sin^{n-1} \theta_n \int_{C_{n-1}} 1 \right) \sin \theta_n \, d\theta_n = \omega_{n-1} \int_0^\pi \sin^n \theta \, d\theta. \tag{2}$$

What is lacking is a general formula for the integral(s)

$$\int_0^\pi \sin^n \theta \, d\theta.$$

Since we have the base cases above, we can assume $n \geq 2$ and write

$$\int_0^\pi \sin^{n-2} \theta (1 - \cos^2 \theta) \, d\theta = \int_0^\pi \sin^{n-2} \theta \, d\theta - \int_0^\pi \cos^2 \theta \sin^{n-2} \theta \, d\theta.$$

The first integral explains why the values take a different form when n is even and when n is odd. In the second integral, we integrate by parts:

$$\int_0^\pi \cos \theta \; \sin^{n-2} \theta \cos \theta \, d\theta = \int_0^\pi \cos \theta \frac{d}{d\theta} \left(\frac{1}{n-1} \sin^{n-1} \theta \right) \, d\theta$$

$$= \frac{\cos \theta}{n-1} \sin^{n-1} \theta \Big|_{\theta=0}^\pi + \int_0^\pi \sin \theta \left(\frac{1}{n-1} \sin^{n-1} \theta \right) \, d\theta$$

$$= \frac{1}{n-1} \int_0^\pi \sin^n \theta \, d\theta.$$

Making this substitution, we get

$$\int_0^\pi \sin^n \theta \, d\theta = \int_0^\pi \sin^{n-2} \theta \, d\theta - \frac{1}{n-1} \int_0^\pi \sin^n \theta \, d\theta$$

or

$$\frac{n}{n-1} \int_0^\pi \sin^n \theta \, d\theta = \int_0^\pi \sin^{n-2} \theta \, d\theta.$$

Starting with the evens

$$\int_0^\pi \sin^2 \theta \, d\theta = \frac{\pi}{2}.$$

7

Therefore,

$$\int_0^\pi \sin^4 \theta \, d\theta = \frac{4-1}{4} \int_0^\pi \sin^{4-2} \theta \, d\theta = \frac{3}{4} \int_0^\pi \sin^2 \theta \, d\theta = \frac{3\pi}{8}.$$

We obtain inductively that for $n = 2k$ even (and $k = 1, 2, 3, \ldots$)

$$\int_0^\pi \sin^{2k} \theta \, d\theta = \frac{\pi}{2^{2k-1}} \begin{pmatrix} 2k - 1 \\ k \end{pmatrix} \qquad (3)$$

where the symbol

$$\begin{pmatrix} 2k - 1 \\ k \end{pmatrix} = \frac{(2k-1)!}{k!(k-1)!}$$

represents the "combination" of $2k - 1$ taken k (at a time). Substituting this formula into (2) when $n = 2k$ is even, we find by another induction that

$$\omega_{2k} = \frac{\pi^k}{k!}. \qquad (4)$$

If we consider $n = 2k+1$ odd for $k = 1, 2, 3, \ldots$, the formulas are (somewhat strikingly) different:

$$\int_0^\pi \sin^{2k+1} \theta \, d\theta = \frac{2^{2k+1}}{k+1} \Bigg/ \begin{pmatrix} 2k + 1 \\ k \end{pmatrix} \qquad (5)$$

and

$$\omega_{2k+1} = 2^{2k+1} \frac{\pi^k k!}{(2k+1)!}. \qquad (6)$$

Notice the value of (3) corresponding to even powers of sine is transcendental with a factor of π while the value of (5) corresponding to odd powers of sine is a rational number. The volumes (4) and (6) are both multiples of a power of π, though the expression looks rather more complicated when the dimension n is odd.

Some elementary calculations with the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

should lead to the unified formula given in the homework assignment.

Exercise 1 Can you make sense of the azimuthal parameterization in calculating the length ω_1 of $B_1(0) \subset \mathbb{R}^1$. What can you say about ω_0?

8

Exercise 2 Find analogues of (3) and (5) for the cosine function:

$$\int_0^\pi \cos^{2k} \theta \, d\theta = \int_0^\pi \sin^{2k} \theta \, d\theta$$

but

$$\int_0^\pi \cos^{2k+1} \theta \, d\theta = 0.$$