Final Assignment (10): Classical Mathematical Methods in Engineering selected solutions

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In this assignment L and T denote positive numbers.

Problem 1 (calculus of variations) This problem is about the motion of a "particle" with position $\mathbf{x} = \mathbf{x}(t)$ moving is three-dimensional space (modeled by) $\mathbb{R}^3$ and having mass $m > 0$. We take as an admissible class

$$\mathcal{A} = \left\{ \mathbf{x} \in C^2([0,T] \to \mathbb{R}^3) : \mathbf{x}(0) = \mathbf{p},\ \mathbf{x}(T) = \mathbf{q} \right\}.$$

Thus, we are considering all the different motions by which this "particle" can move from $\mathbf{p} \in \mathbb{R}^3$ to $\mathbf{q} \in \mathbb{R}^3$ in a given fixed time T.

(a) The **total kinetic energy functional** $\mathcal{K} : \mathcal{A} \to (0, \infty)$ associates to each motion $\mathbf{x} \in \mathcal{A}$ a positive real number

$$\mathcal{K}[\mathbf{x}] = \int_0^T \frac{1}{2}\, m \left| \frac{d\mathbf{x}}{dt} \right|^2 dt.$$

Find the physical dimensions of the total kinetic energy $\mathcal{K}$. Explain how/why these physical dimensions might suggest an "energy of arrangement" like Dirichlet energy.

(b) The **total potential energy functional** $\mathcal{L} : \mathcal{A} \to \mathbb{R}$ associates to each motion a second real number

$$\mathcal{L}[\mathbf{x}] = \int_0^T \Phi(\mathbf{x}(t))\, dt$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is called a (spatially dependent) **force potential**. Determine the physical dimensions of the force potential $[\Phi]$ so that $\mathcal{K}$ and $\mathcal{L}$ have the same physical dimensions.

(c) A motion $\mathbf{x} \in \mathcal{A}$ is called **preferred** if

$$\delta(\mathcal{K} - \mathcal{L})_{\mathbf{x}}[\phi] = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T).$$

Find the ordinary differential equation satisfied by a preferred motion. Hint: Your ordinary differential equation should involve minus the gradient of the force potential

$$-D\Phi(\mathbf{x})$$

which has a special name.

The functional $\mathcal{K} - \mathcal{L}$ giving the difference between the total kinetic and potential energies is called **Hamilton's action functional** or just the **action functional**.

Solution:

(a) Recall energy has dimensions

$$[\text{energy}] = [\text{force}][\text{distance}] = [\text{mass acceleration}]L = \frac{M}{L}T^2\ L = \frac{ML^2}{T^2}.$$

Thus, if an energy like kinetic energy or potential energy which should have dimensions ML^2/T^2 is integrated over a time interval, one obtains a quantity with dimensions

$$\frac{ML^2}{T}.$$

By integrating the kinetic energy of a particular motion over a total time interval one obtains a quantity which may be minimized (or maximimized) by rearranging the "shape" of the motion. Given the kinetic energy associated with a particular function $\mathbf{x}$ which describes a motion from some initial position to a final position, changing or rearranging the motion so that the kinetic energy decreases, i.e., so one moves from the initial position to the final position more slowly but in the same amount of time by taking a more circuitous route, will change/lower the value of the total kinetic energy functional.

(b) The force potential should have also the units of energy:

$$\frac{ML^2}{T^2}.$$

(c) The first variation of the action is

$$\begin{aligned}
\delta(\mathcal{K}-\mathcal{L})_{\mathbf{x}}[\phi] &= \frac{d}{dh}\left(\int_0^T \frac{1}{2}m\left|\frac{d\mathbf{x}}{dt}+h\frac{d\phi}{dt}\right|^2 dt - \int_0^T \Phi(\mathbf{x}+h\phi)\,dt\right)_{h=0} \\
&= \left(\int_0^T m\left(\frac{d\mathbf{x}}{dt}+h\frac{d\phi}{dt}\right)\cdot\frac{d\phi}{dt}\,dt - \int_0^T D\Phi(\mathbf{x}+h\phi)\cdot\phi\,dt\right)_{h=0} \\
&= \int_0^T m\frac{d\mathbf{x}}{dt}\cdot\frac{d\phi}{dt}\,dt - \int_0^T D\Phi(\mathbf{x})\cdot\phi\,dt.
\end{aligned}$$

Note that $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ here is a vector valued function and

$$\frac{d\mathbf{x}}{dt}\cdot\frac{d\phi}{dt} = \sum_{j=1}^n \frac{dx_j}{dt}\,\frac{d\phi_j}{dt},$$

so the first time integral is a sum of integrals

$$\sum_{j=1}^{n} \int_0^T m \frac{dx_j}{dt} \cdot \frac{d\phi_j}{dt}\, dt$$

each of which may be integrated by parts:

$$\begin{aligned}
\sum_{j=1}^{n} \int_0^T m \frac{dx_j}{dt}\ \frac{d\phi_j}{dt}\, dt &= \sum_{j=1}^{n} \left(m\frac{dx_j}{dt}\ \phi_j \Big|_{t=0}^{T} - \int_0^T \frac{d}{dt}\left(m\frac{dx_j}{dt} \right) \phi_j\, dt \right) \\
&= -\sum_{j=1}^{n} \int_0^T \frac{d}{dt}\left(m\frac{dx_j}{dt} \right) \phi_j\, dt \\
&= -\int_0^T \frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right) \cdot \phi\, dt.
\end{aligned}$$

Substituting into the expression for the first variation we have

$$\delta(\mathcal{K} - \mathcal{L})_{\mathbf{x}}[\phi] = \int_0^T \left(-\frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right) - D\Phi(\mathbf{x}) \right) \cdot \phi\, dt.$$

If this quantity vanishes for each vector valued function $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ with $\phi_j \in C_c^\infty(0, T)$ for $j = 1, 2, \ldots, n$, then

$$-\frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right) - D\Phi(\mathbf{x}) = 0$$

or

$$\frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right) = -D\Phi(\mathbf{x}).$$

Notice one can take $\phi_j \equiv 0$ for $j \neq k$ and express the first variation condition as an integral for only the k-th component, and apply the fundamental lemma of the calculus of variations to get the equation for the k-th component in

$$\frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right) = -D\Phi(\mathbf{x}).$$

The special name for negative the spatial gradient of the force potential

$$\mathbf{F} = -D\Phi(\mathbf{x})$$

is “force,” so the Euler-Lagrange equation(s) here are Newton’s equations of motion for a particle/point mass:

$$\mathbf{F} = \frac{d}{dt}\left(m\frac{d\mathbf{x}}{dt} \right).$$

Problem 2 Find the force potential Φ associated with the gravitational force determined by a point mass M located at the origin in $\mathbb{R}^3$. Hint(s): The force on a mass m located at a point $\mathbf{x} \in \mathbb{R}^3$ has magnitude

$$\frac{GMm}{r^2}$$

where $r = |\mathbf{x}|$ is the distance from the location $\mathbf{x}$ to the mass M, and the force is directed toward the origin. The potential energy is obtained by integrating the force against the distance.

Find the equations of motion in the centrally symmetric gravitational field using the calculus of variations/Hamilton's principle from Problem 1.

Solution: One seeks a real valued function of the radius $|\mathbf{x}|$ with negative spatial gradient having magnitude

$$\frac{GMm}{|\mathbf{x}|^2}$$

and radially inward direction. That is,

$$-D\Phi(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Such a function is

$$\Phi(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|}.$$

In fact with this choice

$$\frac{\partial}{\partial x_j}\Phi(\mathbf{x}) = \frac{GMm}{|\mathbf{x}|^3}\, x_j.$$

Problem 3 Consider the following generalization of Newton's second law to a real valued function $w \in C^2(\Omega \times (0,\infty))$: If R is a subregion of Ω, then there is some $\xi \in R$ for which

$$m(R)\; w_{tt}(\xi, t) = \epsilon \int_{\partial R} Dw \cdot N$$

where $m = M(R)$ is the mass of the region $w(R,t) = \{w(\mathbf{x},t) : \mathbf{x} \in R\}$. Assume the mass of the undeformed medium is determined by a constant density ρ_0 and use the divergence theorem to derive the wave equation/operator on $\Omega \subset \mathbb{R}^n$.

Problem 4 (1-D internal oscillations, modeling) For $\alpha > 0$, let $h = h_\alpha : [0, L] \to \mathbb{R}$ by $h(x) = \alpha x$ model a homogeneous internal deformation. Let us call such a function a **homogeneous deformation function**. The potential energy associated with a homogeneous deformation function h is modeled by

$$\frac{\epsilon}{2}\int_0^L (h'(x) - 1)^2\, dx.$$

(a) Two homogeneous deformation functions are said to be **conjugate** if they have the same potential energy. For example, $h_{0.5}$ and $h_{1.5}$ are conjugate. Characterize all conjugate pairs (h_α, h_β) of homogeneous deformation functions.

(b) Given $\omega > 0$, consider the function $w : [0, L] \times \mathbb{R} \to \mathbb{R}$ by

$$w(x,t) = \frac{x}{2}(2 - \cos(\omega t)) \tag{1}$$

(i) Verify w satisfies

$$\begin{cases} w(x,0) = h_{0.5}(x), & 0 < x < L \\ w(x,\pi/\omega) = h_{1.5}(x), & 0 < x < L \\ w(0,t) = 0, & t > 0. \end{cases}$$

(ii) Use mathematical software to animate the motion modeled by w (using time as the animation parameter) for various values of the equilibrium length L and the frequency ω.

(iii) For L fixed, can you find a value for ω corresponding to the "most physically reasonable" motion of the form (1)?

Solution:

(a) The point here is that for each extension with $\alpha > 1$ there should be a compression and an extension with the same potential energy. The expression for the potential energy in terms of $h'_\alpha = \alpha$ is

$$\frac{\epsilon}{2}\int_0^L (\alpha - 1)^2\, dx = \frac{\epsilon}{2}(\alpha - 1)^2 L.$$

If h_α and h_β have the same potential energy, then

$$\frac{\epsilon}{2}(\alpha - 1)^2 L = \frac{\epsilon}{2}(\beta - 1)^2 L.$$

That is,

$$(\alpha-1)^2-(\beta-1)^2=(\alpha-\beta)(\alpha+\beta-2)=0.$$

Assuming $\alpha \neq \beta$, this gives the condition

$$\alpha+\beta=2$$

for conjugate deformations. From the condition, evidently one of α and β must be greater than 1. Let $\alpha > 1$. Then we should have

$$\beta = 2-\alpha.$$

The condition for a deformation to be physical is $h'_\beta > 0$, so this implies also

$$\alpha < 2.$$

In this way, we have the following characterization of conjugate deformations: For each α with $1 < \alpha < 2$ (corresponding to an extension) there is a distinct compression $h_{2-\alpha}$ conjugate to h_α. When $\alpha = 1$, there is no distinct conjugate deformation, and when $\alpha \geq 2$, there is no physical conjugate deformation.

(b) **(i)**

$$w(x,0)=\frac{x}{2}(2-1)=0.5\ x=h_{0.5}(x).$$

$$w(x,\pi/\omega)=\frac{x}{2}(2-(-1))=1.5\ x=h_{1.5}(x).$$

$$w(0,t)=\frac{0}{2}(2-\cos(\omega t))=0.$$

(ii) See the Mathematica notebook `assfinals.nb`.

(iii) One way to approach this question is to plug the formula for w into the wave equation.

$$w_{tt}=\omega^2\ \frac{x}{2}\ \cos\omega t.$$

$$w_{xx}=0.$$

This suggests that this kind of "oscillation" is not a kind of natural or physical oscillation, at least as modeled by the wave equation. Thus, perhaps the answer to which one is led is "no, there is not a particular value of ω corresponding to the physical frequency, because this is not a physical motion."

Alternatively, the motion of Problem 5 below suggests there is a natural frequency associated with this physical problem which does impose a natural frequency $\pi/(2L)$. Thus, one might suggest taking $\omega = \pi/(2L)$ is the most physically reasonable choice.

Problem 5 (1-D internal oscillations) Cconsider the function $w : [0, L] \times \mathbb{R} \to \mathbb{R}$ by

$$w(x,t) = x + 0.5 \cos\left(\frac{\pi t}{2L}\right) \sin\left(\frac{\pi x}{2L}\right). \tag{2}$$

Use mathematical software to animate the motion modeled by w (using time as the animation parameter) for some value of the equilibrium length L.

Problem 6 (1-D internal oscillations, modeling) Let $\alpha > 0$ be given. Consider the initial/boundary value problem

$$\begin{cases} w_{tt} = w_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ w(x,0) = \alpha x, & 0 < x < L \\ w_t(x,0) = 0, & 0 < x < 0 \\ w(0,t) = 0, & t > 0. \end{cases} \tag{3}$$

(a) Determine/figure out the appropriate boundary condition at $x = L$ corresponding to a **free end** for the one-dimensional elastic medium with internal oscillations modeled by the wave operator in (3).

(b) Solve the problem obtained from (3) by appending your condition from part **(a)**. Hint: Subtract the equilibrium solution $h(x) = x$ writing $u = w - x$ and use separation of variables and Fourier series to solve for u.

(c) Make an animation for the motion.

Problem 7 (Problem 6 above) Examine carefully what your solution from part **(b)** of Problem 6 says about what happens at $x = L$ in the parameter space. In particular, compare

$$\frac{\partial w}{\partial x}(L,0) \qquad \text{and} \qquad \frac{\partial w}{\partial x}(L,t) \quad \text{for} \quad t > 0.$$

Does your model predict infinite speed propagation?

Problem 8 (heat equation) Formulate and solve an appropriate initial/boundary value problem modeling heat conduction in a 1-D rod with the temperature $u(0,t) = 0$ fixed, the end at $x = L$ insulated, and with initial temperature distribution

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right).$$

Solution: Here is the initial/boundary value problem:

$$\begin{cases} u_t = u_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = \sin(\pi x/L), & x \in (0,1) \\ u(0,t) = 0, & t > 0 \\ u_x(L,0) = 0, & t > 0. \end{cases}$$

I look for separated variables solutions $u(x,t) = A(x)B(t)$. The PDE gives

$$AB' = A''B$$

or

$$\frac{B'}{B} = \frac{A''}{A} = -\lambda.$$

The boundary conditions give $A(0)B(t) = 0$ and $A'(L)B(t) = 0$ from which we derive $A(0) = 0 = A'(L)$. The Sturm-Liouville problem for $A = A(x)$ has solutions

$$A_j(x) = \sin\left(\frac{(2j+1)\pi}{2L}x\right)$$

with

$$\lambda_j = \frac{(2j+1)^2\pi^2}{4L^2}, \qquad j = 0, 1, 2, 3, \ldots.$$

Thus we seek a superposition

$$u(x,t) = \sum_{j=0}^{\infty} a_j \, e^{-\left(\frac{(2j+1)^2\pi^2}{4L^2} t\right)} \sin\left(\frac{(2j+1)\pi}{2L}x\right).$$

It remains to choose the coefficients a_j to satisfy the initial condition:

$$a_j = \frac{2}{L}\int_0^L \sin\left(\frac{(2j+1)\pi}{2L}x\right) \sin\left(\frac{\pi x}{L}\right) dx = \frac{8(-1)^{j+1}}{(4j^2+4j-3)\pi}.$$

Thus, the solution is given by the series

$$u(x,t) = \sum_{j=0}^{\infty} \frac{8(-1)^{j+1}}{(4j^2+4j-3)\pi} \, e^{-\left(\frac{(2j+1)^2\pi^2}{4L^2} t\right)} \sin\left(\frac{(2j+1)\pi}{2L}x\right).$$

Problem 9 (Wave equation; Haberman sections 7.7-8) Solve the initial/boundary value problem

$$\begin{cases} u_{tt} = \Delta u, & (x,y,t) \in B_1(\mathbf{0}) \times (0,\infty) \\ u(x,y,0) = 1 - (x^2+y^2), & (x,y) \in B_1(\mathbf{0}) \\ u_t(x,y,0) = 0, & (x,y) \in B_1(\mathbf{0}) \\ u(x,y,t) = 0, & (x,y,t) \in \partial B_1(\mathbf{0}) \times (0,\infty) \end{cases} \tag{4}$$

where $B_1(\mathbf{0}) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Hint(s): Change to polar coordinates; use separation of variables.

Solution: Letting $u(x,y,t) = A(x,y)B(t)$, we get

$$AB'' = \Delta A\ B \qquad \text{or} \qquad \frac{B''}{B} = \frac{\Delta A}{A} = -\lambda.$$

For the Laplace operator, we know there is a sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

of eigenvalues which are all positive. We note furthermore that there exist simple eigenfunctions A_j, $j = 1, 2, \ldots$ forming a Basis for $L^2(B_1(\mathbf{0}))$.

Solving $B'' = -\lambda_j B$, we obtain corresponding oscillatory factors

$$B_j(t) = a_j \cos(\mu_j t) + b_j \sin(\mu_j t)$$

with $\mu_j = \sqrt{\lambda_j}$. Denoting the corresponding eigenfunctions for $A = A(x,y)$ by $A_j(x,y)$, we eventually seek a series solution

$$u(x,y,t) = \sum_{j=1}^{\infty} A_j(x,y)(a_j \cos(\mu_j t) + b_j \sin(\mu_j t)).$$

The condition $u_t(x,y,0) = 0$ gives

$$\sum_{j=0}^{\infty} \mu_j b_j A_j(x,y) \equiv 0.$$

Consequently, we conclude $b_j = 0$, $j = 0, 1, 2, 3, \ldots$. It remains to find the functions $A_j(x,y)$ and the coefficients a_j for $j = 0, 1, 2, 3, \ldots$.

We reexpress the boundary value problem for $A = A_j$ in terms of polar coordinates writing

$$v = v_j(r, \theta) = A_j(r\cos\theta, r\sin\theta)$$

for $0 \le r \le a$ and $\theta \in \mathbb{R}$. Differentiating and using the PDE $\Delta A = -\lambda_j A$, we find

$$r^2 v_{rr} + r v_r + v_{\theta\theta} + \lambda_j v = 0.$$

Separating variables again for this PDE we write $v(r, \theta) = C(r)D(\theta)$ and find

$$r^2 C'' D + r C' D + C D'' + \lambda r^2 C D = 0 \qquad \text{or} \qquad \frac{r^2 C'' + r C' + \lambda r^2 C}{C} = -\frac{D''}{D} = \sigma.$$

We have periodic boundary conditions $C(\theta + 2\pi) = C(\theta)$ and $C'(\theta + 2\pi) = C'(\theta)$ to go along with the Sturm-Liouville ODE $D'' = -\kappa D$ to make a familiar (if singular) Sturm-Liouville problem. For this problem we have Sturm-Liouville eigenvalues

$$\sigma_k = k^2, \qquad k = 0, 1, 2, 3, \ldots$$

and corresponding solutions

$$D_k(\theta) = \alpha_k \cos(k\theta) + \beta_k \sin(k\theta).$$

Finally, then we are faced with the ODE

$$r^2 C'' + r C' + (\lambda r^2 - k^2) C = 0$$

with boundary conditions

$$C(0) \text{ exists} \qquad \text{and} \qquad C(1) = 0.$$

Substituting $\xi = r\sqrt{\lambda}$ and setting $\phi(\xi) = C(\xi/\sqrt{\lambda})$ we have

$$\xi^2 \phi'' + \xi \phi' + (\xi^2 - k^2)\phi = 0$$

which is a Bessel ODE of order k with general solution

$$\phi = c J_k(\xi) + d Y_k(\xi)$$

where J_k is the k-th order Bessel function of the first kind and Y_k is the k-th order Bessel function of the second kind. The condition $C(0)$ exists implies $\phi(0)$ exists also and so we must have $d = 0$. The condition $C(1) = 0$ becomes

$$J_k(\sqrt{\lambda}) = 0.$$

This means the positive number $\lambda = \lambda_{kj}$ should satisfy

$$\sqrt{\lambda} = z_{kj}$$

is the j-th positive zero of the order k Bessel function of the first kind. Here $k = 0, 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$. Solving for λ we have,

$$\lambda = \lambda_{kj} = z_{kj}^2.$$

Finally, we obtain

$$C_{kj}(r) = J_k\left(z_{kj}\ r\right).$$

Indexing the coefficients a_j more appropriately as a_{kj} for $k = 0, 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$, the series solution (in polar coordinates) takes the form

$$u(x, y, t) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} a_{kj}(\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)) J_k\left(z_{kj}\ r\right) \cos(z_{kj}\ t).$$

For the initial condition we want

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} a_{kj}(\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)) J_k\left(z_{kj}\ r\right) = 1 - r^2.$$

Notice this initial condition is independent of θ, so $\alpha_k = \beta_k = 0$ for $k = 1, 2, 3, \ldots$ and the condition becomes

$$\sum_{j=1}^{\infty} a_{0j} \alpha_0 J_0(z_{0j}\ r) = 1 - r^2.$$

Evidently we can take $\alpha_0 = 1$, and we need to find the coefficients a_{0j}. Multiplying both sides of the initial condition for the series by $r J_0(z_{0\ell}\ r)$ and integrating we find

$$a_{0\ell} \int_0^1 r[J_0(z_{0\ell}\ r)]^2\, dr = \int_0^1 r(1 - r^2) J_0(z_{0\ell}\ r)\, dr.$$

Thus, the coefficients are

$$a_{0j} = \frac{\displaystyle\int_0^1 r(1 - r^2) J_0(z_{0j}\ r)\, dr}{\displaystyle\int_0^1 r[J_0(z_{0j}\ r)]^2\, dr},$$

and

$$u(x, y, t) = \sum_{j=1}^{\infty} a_{0j} J_k\left(z_{0j}\ \sqrt{x^2 + y^2}\right) \cos(z_{kj}\ t)$$

gives the solution.

Problem 10 (Laplace's equation; Haberman sections 7.7-8) Solve the boundary value problem

$$\begin{cases} \Delta u = 0, & (x,y,z) \in B_1(\mathbf{0}) \times (0,L) \\ u(x,y,0) = 1 - (x^2+y^2), & (x,y) \in B_1(\mathbf{0}) \\ u(x,y,L) = 0, & (x,y) \in B_1(\mathbf{0}) \\ u(x,y,z) = 0, & (x,y,z) \in \partial B_1(\mathbf{0}) \times (0,L) \end{cases} \tag{5}$$

where $B_1(\mathbf{0}) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Hint(s): Change to cylindrical coordinates; use separation of variables.

Solution: This is a static problem in three dimensions. Starting first with a separation $u(x,y,z) = A(x,y)B(z)$ the PDE gives

$$\Delta AB + AB'' = 0 \qquad \text{or} \qquad \frac{\Delta A}{A} = -\frac{B''}{B} = -\lambda.$$

The function $A : B_1(\mathbf{0}) \to \mathbb{R}$ should satisfy the boundary conditions

$$A\Big|_{\partial B_1(\mathbf{0})} = 0.$$

Thus, integrating the PDE $A\Delta A = -\lambda A^2$ we get

$$\begin{aligned} -\lambda \int_{B_1(\mathbf{0})} A^2 &= \int_{B_1(\mathbf{0})} A\Delta A \\ &= \int_{B_1(\mathbf{0})} [\operatorname{div}(A\ DA) - |DA|^2] \\ &= \int_{\partial B_1(\mathbf{0})} A\ DA \cdot \mathbf{n} - \int_{B_1(\mathbf{0})} |DA|^2 \\ &= -\int_{B_1(\mathbf{0})} |DA|^2. \end{aligned}$$

We conclude

$$\lambda = \frac{\displaystyle\int_{B_1(\mathbf{0})} |DA|^2}{\displaystyle\int_{B_1(\mathbf{0})} A^2} > 0.$$

As with the previous problem, we can change to polar coordinates with $v(r,\theta) = C(r)D(\theta) = A(r\cos\theta, r\sin\theta)$. We then get the separation

$$r^2C''D + rC' + CD'' + \lambda r^2 CD = 0 \qquad \text{or} \qquad \frac{r^2C'' + rC' + \lambda r^2 C}{C} = -\frac{D''}{D} = -\sigma$$

exactly as before. Recall that the periodic boundary conditions give $\sigma = \sigma_k = k^2$ for $k = 0, 1, 2, 3, \ldots$. However, we may anticipate at this point that only the constant mode will have a nonzero coefficient. This is because the boundary condition at $z = 0$ is axially symmetric (with no θ dependence), and this is the only inhomogeneous boundary condition. The θ dependence dropped out in the previous problem because the initial condition was axisymmetric. We can also at this point, go ahead and write

$$\lambda = \lambda_{0j} = \lambda_j$$

for the first separation constant though we don't quite know what that (positive) constant is at the moment. Thus, we can take a superposition

$$u(x,y,z) = \sum_{j=1}^{\infty} a_{0j}\ v_{0j}(\sqrt{x^2+y^2})\ B_j(z)$$

where

$$\begin{cases} B_j'' = \lambda B_j, \quad 0 < z < L \\ B_j(L) = 0, \end{cases}$$

and $C = C_{0j}$ with

$$\begin{cases} r^2C'' + rC' + \lambda r^2 C = 0, \quad 0 < r < 1 \\ C(0) \text{ exists}, \\ C(1) = 0. \end{cases}$$

Writing $C(r) = \phi(r\sqrt{\lambda_j})$ the second problem becomes

$$\begin{cases} \xi^2\phi'' + \xi\phi' + \xi^2\phi = 0, \quad 0 < \xi < \sqrt{\lambda_j} \\ \phi(0) \text{ exists}, \\ \phi(\sqrt{\lambda}) = 0. \end{cases}$$

The ODE here is a zero order Bessel equation and the bounded value at zero means the solution should be (a multiple of) the zero order Bessel function $\phi = J_0$. The condition at $\xi = \sqrt{\lambda}$ implies $\lambda = \lambda_j = z_{0j}^2$ is a square of a zero of J_0 as in the solution of the wave equation in Problem 9 above. The function $C = C_{0j} = C_j$ takes the form

$$C_j(r) = J_0(z_{0j} r) = J_0(z_{0j}\sqrt{x^2+y^2}).$$

The problem for $B = B_j$ now becomes

$$\begin{cases} B_j'' = z_{0j}^2 B_j, & 0 < z < L \\ B_j(L) = 0, \end{cases}$$

A nice "trick" at this point is to observe that one can take

$$\left\{ \cosh(z_{0j}(z-L)), \sinh(z_{0j}(z-L)) \right\}$$

as the basis of solutions for the ODE $B'' = z_{0j}^2 B$ so the general solution is

$$B_j = b_j \cosh(z_{0j}(z-L)) + a_j \sinh(z_{0j}(z-L)).$$

Since $B_j(L) = 0$, we get $b_j = 0$, and the superposition becomes

$$u(x,y,z) = \sum_{j=1}^{\infty} a_j \; J_0(z_{0j}\sqrt{x^2+y^2}) \; \sinh(z_{0j}(z-L)).$$

The last boundary condition at $z = 0$ requires

$$-\sum_{j=1}^{\infty} a_j \; J_0(z_{0j} r) \; \sinh(z_{0j} L) = 1 - r^2.$$

Thus,

$$a_j = -\frac{\displaystyle\int_0^1 r(1-r^2) J_0(z_{0j} r)\, dr}{\sinh(z_{0j} L) \displaystyle\int_0^1 r[J_0(z_{0j} r)]^2\, dr.}$$

These integrals take a little while to compute numerically. Some such computations are included in `assfinals-26.nb` for Problem 9 above, and also these coefficients appear to be correct or Problem 10.

As a final note, if one has an inhomogeneous lateral boundary condition for this problem, then the solution involves **modified Bessel functions** K_j and I_j which are also available in Mathematica.