Final Assignment (10): Classical Mathematical Methods in Engineering
under construction
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(more problems coming later)
Due Wednesday May 6, 2024

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Problem 1 (calculus of variations) This problem is about the motion of a "particle" with position $\mathbf{x} = \mathbf{x}(t)$ moving is three-dimensional space (modeled by) $\mathbb{R}^3$ and having mass $m > 0$. We take as an admissible class

$$\mathcal{A} = \left\{ \mathbf{x} \in C^2([0,T] \to \mathbb{R}^3) : \mathbf{x}(0) = \mathbf{p},\ \mathbf{x}(T) = \mathbf{q} \right\}.$$

Thus, we are considering all the different motions by which this "particle" can move from $\mathbf{p} \in \mathbb{R}^3$ to $\mathbf{q} \in \mathbb{R}^3$ in a given fixed time T.

(a) The **total kinetic energy functional** $\mathcal{K} : \mathcal{A} \to (0, \infty)$ associates to each motion $\mathbf{x} \in \mathcal{A}$ a positive real number

$$\mathcal{K}[\mathbf{x}] = \int_0^T \frac{1}{2}\, m \left| \frac{d\mathbf{x}}{dt} \right|^2 dt.$$

Find the physical dimensions of the total kinetic energy $\mathcal{K}$. Explain how/why these physical dimensions might suggest an "energy of arrangement" like Dirichlet energy.

(b) The **total potential energy functional** $\mathcal{L} : \mathcal{A} \to \mathbb{R}$ associates to each motion a second real number

$$\mathcal{L}[\mathbf{x}] = \int_0^T \Phi(\mathbf{x}(t))\, dt$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is called a (spatially dependent) **force potential**. Determine the physical dimensions of the force potential $[\Phi]$ so that $\mathcal{K}$ and $\mathcal{L}$ have the same physical dimensions.

(c) A motion $\mathbf{x} \in \mathcal{A}$ is called **preferred** if

$$\delta(\mathcal{K} - \mathcal{L})_{\mathbf{x}}[\phi] = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T).$$

Find the ordinary differential equation satisfied by a preferred motion. Hint: Your ordinary differential equation should involve minus the gradient of the force potential

$$-D\Phi(\mathbf{x})$$

which has a special name.

The functional $\mathcal{K} - \mathcal{L}$ giving the difference between the total kinetic and potential energies is called **Hamilton's action functional** or just the **action functional**.

Problem 2 Find the force potential Φ associated with the gravitational force determined by a point mass M located at the origin in $\mathbb{R}^3$. Hint(s): The force on a mass m located at a point $\mathbf{x} \in \mathbb{R}^3$ has magnitude

$$\frac{GMm}{r^2}$$

where $r = |\mathbf{x}|$ is the distance from the location $\mathbf{x}$ to the mass M, and the force is directed toward the origin. The potential energy is obtained by integrating the force against the distance.

Find the equations of motion in the centrally symmetric gravitational field using the calculus of variations/Hamilton's principle from Problem 1.

Problem 3 Consider the following generalization of Newton's second law to a real valued function $w \in C^2(\Omega \times (0, \infty))$: If R is a subregion of Ω, then there is some $\xi \in R$ for which

$$m(R)\ w_{tt}(\xi, t) = \epsilon \int_{\partial R} Dw \cdot N$$

where $m = M(R)$ is the mass of the region $w(R, t) = \{w(\mathbf{x}, t) : \mathbf{x} \in R\}$. Assume the mass of the undeformed medium is determined by a constant density ρ_0 and use the divergence theorem to derive the wave equation/operator on $\Omega \subset \mathbb{R}^n$.