Final Assignment (10): Classical Mathematical Methods in Engineering Due Thursday December 12, 2024

John McCuan

Problem 1 Assume $f \in \mathfrak{L}^1(-L, L)$ and the Fourier series

$$a_0 + \sum_{j=1}^{\infty} a_j \cos\left(\frac{j\pi}{L}\, x\right) + \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\pi}{L}\, x\right)$$

has partial sums converging to f in $\mathfrak{L}^1(-L, L)$. Assume termwise integration holds so that

$$\int_{(-L,x)} f = a_0(x+L) + \sum_{j=1}^{\infty} \frac{La_j}{j\pi} \sin\left(\frac{j\pi}{L}\, x\right) + \sum_{j=1}^{\infty} \frac{Lb_j}{j\pi} \left[\cos\left(\frac{j\pi}{L}\, x\right) - (-1)^j\right]$$

holds pointwise for any function and series satisfying these conditions.

(a) Express

$$g(x) = f(x) \sin\left(\frac{k\pi}{L}\, x\right)$$

as a Fourier series in $\mathfrak{L}^1(-L, L)$ satisfying the conditions required for the assumption on termwise integration.

(b) Derive from termwise integration of the series for g obtained in part **(a)** a formula for the coefficient b_j, $j = 1, 2, 3, \ldots$.

Problem 2 (Haberman 1.4.4) Assume heat conduction is modeled in a thin metal rod by

$$u_t = (ku_x)_x \qquad \text{on} \qquad (0, \ell) \times (0, \infty)$$

where $k = k(x)$ depends on position. If both ends of the rod are modeled as insulated, show the total heat energy in the rod must be constant (as a function of time).

Problem 3 (Haberman 1.4.6) If heat conduction in a thin metal rod is modeled by the forced 1-D heat equation with nonzero constant source term Q, and both ends are modeled as insulated, prove there can be no equilibrium solution

$$U(x) = \lim_{t \nearrow \infty} u(x,t).$$

Problem 4 Consider the initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & (x,y,t) \in R \times (0,\infty) \\ u(x,y,0) = u_0, & (x,y) \in R \\ u(x,y,t) = 0, & (x,y,t) \in \partial R \times (0,\infty) \end{cases}$$

for the 2-D heat equation where $R = (0,4) \times (0,2)$ is a rectangular spatial domain in $\mathbb{R}^2$ and

$$u_0(x,y) = 2 - \max\{|x-2|, 2|y-1|\}.$$

(a) Plot the graph of u_0 by hand.

(b) Solve the problem using separation of variables and Fourier series expansion.

(c) Animate the solution using mathematical software with the time t as an animation parameter.

Problem 5 Let $\Phi : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ denote the fundamental solution of the 1-D heat equation. See Problem 8 and Problem 9 of Assignment 7.

Given $u_0 : \mathbb{R} \to \mathbb{R}$ with $u \in C^0(\mathbb{R})$, the function

$$u(x,t) = \int_{\xi \in \mathbb{R}} \Phi(x-\xi,t)\; u_0(\xi)$$

is called the **spatial convolution** of the fundamental solution with u_0. Show that this spatial convolution satisfies the initial value problem

$$\begin{cases} u_t = u_{xx} & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

for the heat equation on the whole real line.

Problem 6 (length measures cannot measure all sets) Complete the steps outlined below[1] in showing it is impossible to have a translation invariant length measure on the interval $[0,1)$ with domain the collection $\wp([0,1))$ of all subsets of $[0,1)$.

The argument is by contradiction. Assume by way of contradiction

$$\mu : \wp([0,1)) \to [0,1] \tag{1}$$

is a function having the following properties:

(i) (countable additivity) If $\{A_j\}_{j=1}^{\infty} \subset \wp([0,1))$ is a countable collection of disjoint sets, i.e., $A_i \cap A_j = \phi$ if $i \neq j$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

(L) If I is any interval in $[0,1)$, meaning I has one of the following forms:

$$\begin{aligned} (a,b) &= \{x : a < x < b\} && \text{for some } a,b \in [0,1) \text{ with } a < b, \\ [a,b) &= \{x : a \leq x < b\} && \text{for some } a,b \in [0,1) \text{ with } a < b, \\ (a,b] &= \{x : a < x \leq b\} && \text{for some } a,b \in [0,1) \text{ with } a < b, \text{ or} \\ [a,b] &= \{x : a \leq x \leq b\} && \text{for some } a,b \in [0,1) \text{ with } a \leq b, \end{aligned}$$

then $\mu(I) = \text{length}(I) = b - a$.

(T) If $A \subset [0,1)$ and $t \in \mathbb{R}$ and $\{x + t : x \in A\} \subset [0,1)$, then

$$\mu(\{x + t : x \in A\}) = \mu(A).$$

The property **(i)** of countable additivity is essentially what makes the function μ a **measure**.[2] A measure satisfying **(L)** is said to be a **length measure**. A measure satisfying **(T)** is said to be **translation invariant**.

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[1]This material is from the book *Real Analysis* by Halsey Royden (1928–1923).

[2]Technically, the real definition of a measure is somewhat more complicated. First of all the domain of a measure is usually taken to be an arbitrary **sigma algebra** of subsets in $\wp([0,1)]$, or in $\wp(X)$ where X is the set whose subsets are being measured. So for a proper definition, one should define the notion of a sigma algebra first. In our case, we are using $\wp([0,1))$ as the sigma algebra, and the power set of any set is always a sigma algebra. Also, in general if μ is allowed to take non-negative extended real values in $[0,\infty]$, then the condition **(ii)** $\mu(\phi) = 0$ is usually included. If you know some set has finite measure, you know μ takes non-negative values, and you know μ is countably additive, then you can prove $\mu(\phi) = 0$.

(a) An **equivalence relation** on a set S is any subset R of $S \times S$ for which the following hold

(i) $(x, x) \in R$ for all $x \in S$,

(ii) If $(x, y) \in R$, then $(y, x) \in R$, and

(iii) If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Property **(i)** is called the **reflexive** property and is usually expressed by writing $x \sim x$, where the equivalence relation is informally represented by the notation "$\sim$." Similarly, an equivalence relation is said to be **symmetric** if **(ii)** holds, and this is informally expressed by writing

$$x \sim y \qquad \Longrightarrow \qquad y \sim x.$$

The third property is called the **transitive** property:

$$x \sim y \quad \text{and} \quad y \sim z \qquad \Longrightarrow \qquad x \sim z.$$

Most of the time when you use the symbol "=" in mathematics, it is denoting some equivalence relation.

Show that any time one has an equivalence relation "$\sim$" on a set S, then the collection

$$\mathcal{P} = \big\{\{y \in S : y \sim x\} : x \in S\big\}$$

is a **partition** of S. Each set $A_x = \{y \in S : y \sim x\}$ is called the **equivalence class** of $x \in S$, and what you need to show is that either two equivalence classes A_x and A_w are disjoint, i.e., $A_x \cap A_w = \phi$, or identical, i.e., $A_x = A_w$. Hint: Remember that in order to show two sets are equal, you need to show each is a subset of the other.

(b) (rational equivalence) Let $\mathbb{Q}$ denote the **rational numbers**

$$\mathbb{Q} = \left\{\frac{m}{n} : n \in \mathbb{N} = \{1, 2, 3, \ldots\} \text{ and } m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}\right\}.$$

Show $x \sim_{\mathbb{Q}} y$ if $x - y \in \mathbb{Q}$ defines an equivalence relation on $[0, 1)$.

As a consequence of parts **(b)** and **(c)** above, the equivalence classes

$$\big\{A_x = \{y \in [0, 1) : y \sim_{\mathbb{Q}} x\} : x \in [0, 1)\big\},$$

where "$\sim_{\mathbb{Q}}$" represents rational equivalence, are a partition of $[0,1)$.

Of course, it may be the case that $A_x = A_y$ for elements $x, y \in [0,1)$ with $x \neq y$. In the application below, however, we use a particular index set $J \subset [0,1)$ for which

$$\big\{A_x : x \in [0,1)\big\} = \{A_x : x \in J\}$$

and $A_x = A_y$ for $x, y \in J$ implies $x = y$. That is to say, the set $J \subset [0,1)$ contains exactly one element from each equivalence class.[3]

(c) (mod 1 addition) The function $m : [0,1) \times [0,1) \to [0,1)$ given by

$$m(x,y) = \begin{cases} x+y, & \text{if } x+y < 1 \\ x+y-1, & \text{if } x+y \geq 1 \end{cases}$$

is called **mod** 1 **addition**. Recall that the rational numbers $\mathbb{Q}$ are countable. This means there is a bijection $r : \mathbb{N} \to \mathbb{Q}$. Let us denote the image $r(j)$ of each natural number j under this bijection is denoted by r_j so that

$$\mathbb{Q} = \{r_j\}_{j=1}^{\infty}.$$

You should now think: r_1 is the first rational number, r_2 is the second rational number, and so on.

For each $j = 1, 2, 3, \ldots$, consider the "r_j shuffle" of J defined by

$$E_j = \{m(x, r_j) : x \in J\}.$$

(i) Draw a picture of the set E_j. (You'll have to be creative about how to illustrate/draw the set J because no one knows what J actually looks like.)

(ii) Use translation invariance to show $\mu(E_i) = \mu(E_j)$ for every $i, j \in \mathbb{N}$.

(iii) Show $E_i \cap E_j = \phi$ if $i \neq j$.

(iv) Show

$$\bigcup_{j=1}^{\infty} E_j = [0,1).$$

Hint: If $x \in [0,1)$, there is some $x_0 \in J$ for which $A_{x_0} = A_x$. That is, $x - x_0 \in \mathbb{Q}$.

[3]Technically, the existence of this set J follows from an application of the **axiom of choice**.

(d) As a consequence of **(c)(iii)** and **(c)(iv)** the collection $\{E_j\}_{j=1}^{\infty}$ is a countable partition of $[0, 1)$. Also, by **(c)(ii)** each set E_j has the same measure. Use the countable additivity from your definition of (abstract) measure to obtain a contradiction showing it is impossible to measure all subsets of $[0, 1)$ with a length measure.

Note: The existence of non-measurable sets may seem like a mathematical curiosity, and from the point of view of applications/engineering it may well be. One can argue that most sets used in applications are measurable. On the other hand, the existence of non-measurable sets has some grave consequences for the Euclidean spaces $\mathbb{R}^n$ which are used pretty much universally in the mathematical modeling of engineering. One such consequence is called the **Banach-Tarski paradox** which applies to $\mathbb{R}^3$: There exist five geometrically congruent sets A_1, A_2, A_3, A_4, and A_5 and five rigid motions ρ_1, ρ_2, ρ_3, ρ_4 and ρ_5 with the following properties:

(i) $A_i \cap A_j = \phi$, that is $A_1, A_2, \ldots, A_5$ are disjoint sets.

(ii) The union of $A_1, A_2, \ldots, A_5$ is a unit ball, that is for example,

$$\bigcup_{j=1}^{5} A_j = B_1(\mathbf{0}) = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}.$$

(ii) The sets $A_1, A_2, \ldots, A_5$ can be rigidly moved around to form two unit balls:

$$\bigcup_{j=1}^{5} \rho_j(A_j) = B_1(\mathbf{0}) \cup B_1(2, 0, 0)$$
$$= \{(x, y, z) : x^2 + y^2 + z^2 < 1\} \cup \{(x, y, z) : (x-2)^2 + y^2 + z^2 < 1\}.$$

A **rigid motion** ρ is a function $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ which is a composition of a rotation and a translation. One way to say two pices A_i and A_j are **congruent** is to say there exists a rigid motion with

$$\rho(A_i)\{\rho(\mathbf{x}) : \mathbf{x} \in A_i\} = A_j.$$