Assignment 9: Wave Equation (review)
Due Wednesday April 22, 2026

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For the problems below L denotes a positive real number. By the **wave equation** we mean the second order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

for a real valued function $u \in C^2(\Omega \times [0, T))$ of n spatial variables and one time variable, that is the PDE $u_{tt} = \Delta u$.

Problems 1-3 are precisely the same as Problems 8-10 of Assignment 8. You do not need to do them again for this assignment. You can just copy your answers from what you turned in on Assignment 8, but you may want to review these problems as Problem 4 is the final problem in this series of problems. Also, it may not hurt you to do them (again).

Problem 1 (factoring the wave operator; same as Problem 8 of Assignment 8) The **wave operator** is also called the d'Alembertian, and it is denoted by $\Box : C^2(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ with

$$\Box u = u_{tt} - u_{xx}.$$

Like the Laplacian $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $\Delta u = u_{xx}$ and the heat operator $L : C^2(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ by $Lu = u_t - u_{xx}$, the d'Alembertian is a second order partial differential operator.

Let $T : C^1(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ be the first order transport operator given by

$$Tu = u_t - u_x.$$

(a) Find a first order linear operator $S : C^1(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ for which

$$\Box u = S \circ T.$$

(b) Compute $T \circ S$.

(c) An operator $A : V \to W$ defined on a vector space V and taking values in a vector space W is **linear** if

$$A(cu) = cAu \qquad \text{for every } c \in \mathbb{R} \text{ and } u \in V$$

and

$$A(u+v) = Au + Av \qquad \text{for every } u, v \in V.$$

(i) Show the wave operator is linear on $C^2(\mathbb{R} \times (0,\infty))$.

(ii) Show the operator T is linear on $C^1(\mathbb{R} \times (0,\infty))$.

(iii) Show the Laplace operator Δ is linear on $C^2((a,b))$ where $a, b \in \mathbb{R}$ with $a < b$.

(iv) Show the heat operator L is linear on $C^2((a,b) \times (0,\infty))$ where $a, b \in \mathbb{R}$ with $a < b$.

Problem 2 (The wave equation on all of $\mathbb{R}$; same as Problem 9 of Assignment 8) Consider the initial value problem (IVP) for the wave equation:

$$\begin{cases} u_{tt} = u_{xx}, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ u_t(x, 0) = 0, & x \in \mathbb{R} \end{cases} \tag{1}$$

where u_0 is a given function with $u_0 \in C^2(\mathbb{R})$. Assuming $u \in C^2(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ is a solution of (1) and $w = Tu \in C^1(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ where S and T are the factor operators from Problem 1, find an appropriate initial value problem for a transport equation satisfied by w.

In particular, find an appropriate initial value w_0 for the problem

$$\begin{cases} Sw = 0, & \text{on } \mathbb{R} \times (0, \infty) \\ w(x, 0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Problem 3 (first order linear equation; same as Problem 10 of Assignment 8) Consider the problem (2) with the initial condition you found in Problem 2. Solve that problem by completing the following steps:

(a) Consider a parameterized path $\gamma : [0, \infty) \to \mathbb{R} \times [0, \infty)$ given by $\gamma(t) = (\xi(t), t)$ for some real valued spatial function $x = \xi(t)$. Use the chain rule to compute

$$\frac{d}{dt} w \circ \gamma(t). \tag{3}$$

(b) Compare your result from (3) to the PDE from (2). Given any initial starting point $x_0 \in \mathbb{R}$ find an appropriate ODE for $\xi : [0, \infty) \to \mathbb{R}$ based on your comparison, and solve the ODE with the initial condition $\xi(0) = x_0$.

(c) With your solution for ξ from part **(b)** which should depend on x_0, consider for an arbitrary point $(x, t) \in \mathbb{R} \times (0, \infty)$ the equation

$$\gamma(t) = (x, t). \tag{4}$$

Choose x_0 so that (4) is satisfied.

(d) If you made the correct choice of ODE in part **(b)** you should now know the value of the quantity in (3), which should tell you

$$w(\xi(t), t) = w_0(x_0).$$

If you made the correct choice of w_0 in Problem 2 you should now know the solution of the problem (2) in terms of u_0 from the IVP in Problem 2.

Problem 4 (another first order linear equation) Consider the inhomogeneous (forced) initial value problem for $u \in C^1(\mathbb{R} \times [0, \infty))$

$$\begin{cases} u_t - u_x = w, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{5}$$

where $u_0, w \in C^0(\mathbb{R})$ are given initial and spatial forcing functions.

(a) Consider a propagating curve $\gamma(t) = (\xi(t), t)$ with $\xi(0) = x_0 \in \mathbb{R}$. Calculate

$$\frac{d}{dt} u \circ \gamma(t) \tag{6}$$

and pose an appropriate ODE for ξ based on comparison with the operator $Tu = u_t - u_x$ in the PDE.

(b) Draw a picture of the curve parameterized by γ in $\mathbb{R} \times [0, \infty)$.

(c) How would you characterize the propagation of x_0 induced by γ? (Give speed and direction.)

(d) Derive an ODE for $u \circ \gamma$ based on your work above and computation of the derivative in (6).

(e) Couple your ODE from part **(d)** with an appropriate initial condition to find a formula for $u \circ \gamma(t)$ as a function of x_0. Be careful with the argument of w.

(f) Solve the equation $\gamma(t) = (x, t)$ for the starting point x_0.

(g) Substitute (x, t) in for $\gamma(t)$ along with the value you found for x_0 in part **(f)** into the formula you found in part **(e)** to solve the problem (5).

Solution:

(a)

$$\frac{d}{dt} u \circ \gamma(t) = u_x \, \xi' + u_t.$$

If we take

$$\begin{cases} \xi' = -1, & t > 0, \\ \xi(0) = x_0, \end{cases}$$

then $(u \circ \gamma)' = u_x \, \xi' + u_t = -u_x + u_t$ matches the operator $Tu = u_t - u_x$ in the PDE.

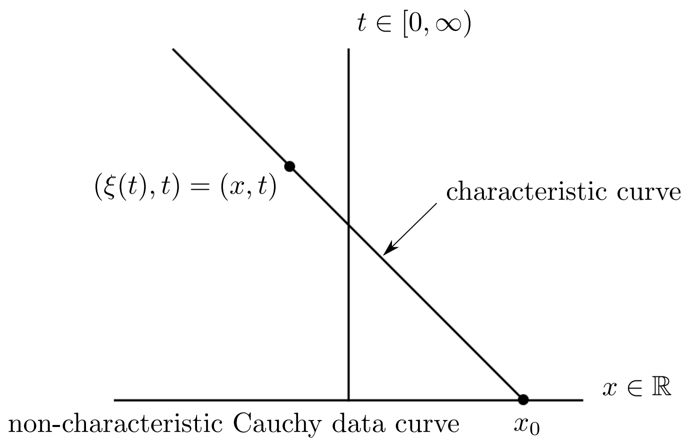

Figure 1: Curve parameterized by γ in $\mathbb{R} \times [0, \infty)$.

(b) Here is a picture of the solution $\gamma(t) = (\xi(t), t)$:

(c) The position x_0 propogates to the left (or "back" or "downward") at unit speed.

(d) The ODE for $u \circ \gamma$ is

$$\frac{d}{dt}(u \circ \gamma)(\xi(t), t) = w(\xi(t)).$$

More generally, one could take a time dependent forcing function $w \in C^0(\mathbb{R} \times [0, \infty))$ and write

$$\frac{d}{dt}(u \circ \gamma)(t) = w(\xi(t), t),$$

but as the problem is phrased the first ODE with only spatial dependence in w is what is suggested.

(e) The initial condition for this problem would be

$$u \circ \gamma(0) = u(x_0, 0) = u_0(x_0).$$

The solution is

$$u \circ \gamma(t) = u_0(x_0) + \int_0^t w(\xi(\tau))\, d\tau.$$

The more general problem is also easy to solve:

$$u \circ \gamma(t) = u_0(x_0) + \int_0^t w(\xi(\tau), \tau)\, d\tau.$$

(f) If $\gamma(t) = (x_0 - t, t) = (x, t)$, then $x_0 = x + t$.

(g) If $(x, t) \in \mathbb{R} \times (0, \infty)$ is given and $x_0 = x + t$, then

$$u(x,t) = u \circ \gamma(t) = u_0(x+t) + \int_0^t w(x+t-\tau)\, d\tau.$$

You can see this solves the problem since we have $u(x, 0) = u_0(x)$,

$$u_x(x,t) = u_0'(x+t) + \int_0^t w'(x+t-\tau)\, d\tau$$

and

$$u_t(x,t) = u_0'(x+t) + w(x) + \int_0^t w'(x+t-\tau)\, d\tau.$$

Note, there is no essential difference if w is allowed to depend on t as well:

$$u(x,t) = u \circ \gamma(t) = u_0(x+t) + \int_0^t w(x+t-\tau, \tau)\, d\tau.$$

$$u_x(x,t) = u_0'(x+t) + \int_0^t w_x(x+t-\tau, \tau)\, d\tau$$

and

$$u_t(x,t) = u_0'(x+t) + w(x) + \int_0^t w_x(x+t-\tau, \tau)\, d\tau.$$

Problem 5 (Problem 4 above; finite propagation speed of a signal) Take the specific choice(s)

$$u_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad w \equiv 0$$

in the formula you obtained in part **(g)** of Problem 4.

(a) Plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(b) If you think of the behavior of u_0 on its support as a "signal," how long does it take for information from the signal to be communicated at $x = -10$? At what time has the signal passed $x = -10$?

Problems 6-8: Oscillators

Problems 6-8 suggest a comparison between the modeling of a mass on a spring using ODEs and the modeling of an internally deformed spring (or more properly thin/one-dimensional elastic medium) with internal mass using the wave equation.

Recall the modeling of a harmonic oscillator (mass on a spring) associated with the IVP

$$\begin{cases} m\ddot{x} = -k(x - L), & t \in \mathbb{R} \\ x(0) = x_0, \\ \dot{x}(0) = v_0. \end{cases} \tag{7}$$

Given an initial position x_0 and initial velocity v_0 for the mass m on a (massless) spring with Hooke's constant k, a nonnegative number modeling the **total energy of the initial configuration** is given by

$$T_0 = \frac{1}{2} m v_0^2 + \frac{1}{2} k (x_0 - L)^2.$$

Problem 6 (energy)

(a) Draw a picture of the physical system modelted by (7).

(b) Evaluate the integral

$$\int_0^x k(\xi - L)\, d\xi$$

and explain why this value is used to model the potential energy for a harmonic oscillator. Hint: potential energy equals "force times distance."

(c) Let $T = T(t)$ denote the sum of the kinetic energy $m\dot{x}^2/2$ and the potential energy associated at each time t with a solution of (7). Show T is (the) constant T_0.

(d) How would this discussion change if (linear) damping were incorporated in the model?

Problem 7 (internal oscillations) Assume an elastic medium with elasticity ϵ and constant lineal density (at equilibrium) ρ_0 is modeled using the initial/boundary value problem

$$\begin{cases} \rho_0 w_{tt} = \epsilon w_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ w(x,0) = w_0(x), & x \in (0,L) \\ w_t(x,0) = v_0(x), & x \in (0,L) \\ w(0,t) = 0, & t > 0 \\ w(L,t) = L, & t > 0. \end{cases} \tag{8}$$

(a) Solve this problem using separation of variables and Fourier series. Hint(s): Set $u = w - x$ and solve the problem for u with homogeneous boundary values. Your answer should involve coefficients depending on the functions w_0 and v_0.

(b) Let α and β be numbers with $1 < \alpha < 2$ and $\beta \geq 0$. Take

$$w_0(x) = \begin{cases} \alpha x, & 0 < x < L/2 \\ (\alpha - 1)L + (2 - \alpha)x, & L/2 < x < L \end{cases} \qquad \text{and} \qquad v_0(x) = \beta \sin \frac{j\pi}{L} x$$

and evaluate the coefficients you found in part **(a)**.

(c) Animate the internal oscillations modeled here (with time as the animation parameter) for specific numerical values of L, ρ_0, ϵ, α and β.

(i) What relations must be maintained among the parameters in order to preserve the modeling assumption $w_x(x,t) > 0$ for all $(x,t) \in (0,L) \times (0,\infty)$?

(ii) If $\beta = 0$, do the relations you found in part **(i)** hold?

(iii) Do you see specific (positive) times t for which

$$w_t(x,t) \equiv 0, \qquad 0 < x < L \ ?$$

What is the frequency associated with these times? (You may wish to restrict attention to $\beta = 0$ here.)

Solution:

(a) If $u = w - x$, then

$$\begin{cases} \rho_0 u_{tt} = \epsilon u_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = w_0(x) - x, & x \in (0,L) \\ u_t(x,0) = v_0(x), & x \in (0,L) \\ u(0,t) = 0, & t > 0 \\ u(L,t) = 0, & t > 0. \end{cases}$$

Separated variables solutions (setting aside the initial values) might have the form

$$u(x,t) = A(x)B(t)$$

and satisfy

$$\rho_0 AB'' = \epsilon A'' B \qquad \text{or} \qquad \rho_0 \frac{B''}{B} = \epsilon \frac{A''}{A} = -\lambda$$

with $A(0) = A(L) = 0$ as regular Sturm-Liouville boundary values. We obtain for $A = A(x)$

$$A_j(x) = \sin\left(\frac{j\pi}{L}x\right) \qquad \text{with} \qquad \lambda_j = \epsilon \frac{j^2\pi^2}{L^2} \quad \text{for } j = 1, 2, 3, \ldots.$$

Corresponding to each λ_j we have oscillations in time with

$$B_j(t) = a_j \cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{j\pi}{L}t\right) + b_j \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{j\pi}{L}t\right)$$

solving

$$\rho_0 B_j'' = -\epsilon\ \frac{j^2\pi^2}{L^2}\ B_j = -\lambda_j\ B_j.$$

Thus, we look for a superposition

$$u(x,t) = \sum_{j=1}^{\infty} \left(a_j \cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{j\pi}{L}t\right) + b_j \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{j\pi}{L}t\right)\right) \sin\left(\frac{j\pi}{L}x\right).$$

Then

$$u(x,0) = \sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi}{L}x\right)$$

and we need

$$u_0(x) = \sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi}{L}x\right)$$

where $u_0 = w_0 - x$. It follows that

$$a_j = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{j\pi}{L}x\right)\ dx.$$

Next,

$$u_t(x,0) = \sqrt{\frac{\epsilon}{\rho_0}}\frac{\pi}{L}\sum_{j=1}^{\infty} j b_j \sin\left(\frac{j\pi}{L}x\right).$$

So we need

$$v_0(x) = \sqrt{\frac{\epsilon}{\rho_0}}\frac{\pi}{L}\sum_{j=1}^{\infty} j b_j \sin\left(\frac{j\pi}{L}x\right)$$

and

$$b_j = \sqrt{\frac{\rho_0}{\epsilon}}\frac{2}{j\pi}\int_0^L v_0(x)\sin\left(\frac{j\pi}{L}x\right)\,dx.$$

With these coefficients to determine u, we should have a solution $w = u + x$.

(b)

$$a_j = \frac{4L(\alpha-1)}{\pi^2 j^2}\sin\left(\frac{j\pi}{2}\right).$$

This means all the even indexed coefficients vanish and

$$c_k = a_{2k+1} = \frac{4L(\alpha-1)}{\pi^2(2k+1)^2}(-1)^k \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

$b_j = 0$ for $j = 2, 3, 4, \ldots$ and

$$b_1 = \sqrt{\frac{\rho_0}{\epsilon}}\,\frac{\beta L}{\pi}.$$

Thus, in this case we can write

$$w(x,t) = x + \frac{4L(\alpha-1)}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\,\frac{(2k+1)\pi}{L}t\right)\sin\left(\frac{(2k+1)\pi}{L}x\right)$$
$$+\sqrt{\frac{\rho_0}{\epsilon}}\,\frac{\beta L}{\pi}\sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\,\frac{\pi}{L}t\right)\sin\left(\frac{\pi}{L}x\right).$$

(c) See the Mathematica notebook `ass9prob7-26.nb` where the case $\rho_0/\epsilon = 1 = L$ is considered for various values of α and β.

(i) The basic idea here is that when $\alpha = 2$ and the initial wave form already has $w_0(x) = 0$ for $L/2 < x < L$, then adding any additional initial velocity to the right will definitely violate the condition $w_x > 0$. Thus, we certainly need $\beta = 0$ when $\alpha = 2$.

For smaller α with $1 < \alpha < 2$ there should be some (positive) maximum β for which the oscillation is physically reasonable with maximum compression $w_x = 0$ somewhere and at some time. Checking this assumption with the animations for $\alpha = 1$ with the equilibrium initial wave form leads to the conclusion $\beta = 1$. This is verified analytically below.

Further checking suggests the "natural" condition (9) below.

Differentiating the expression for w given at the end of the solution of part **(b)** above,

$$w_x = 1 + \frac{4(\alpha-1)}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\, \frac{(2k+1)\pi}{L} t\right) \cos\left(\frac{(2k+1)\pi}{L} x\right) + \beta \sqrt{\frac{\rho_0}{\epsilon}}\, \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\, \frac{\pi}{L} t\right) \cos\left(\frac{\pi}{L} x\right).$$

If this quantity is positive, then

$$-\beta \sqrt{\frac{\rho_0}{\epsilon}}\, \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\, \frac{\pi}{L} t\right) \cos\left(\frac{\pi}{L} x\right) < 1 + \frac{4(\alpha-1)}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\, \frac{(2k+1)\pi}{L} t\right) \cos\left(\frac{(2k+1)\pi}{L} x\right).$$

In principle, we could take this as a necessary relation among the parameters, but the relation must hold for all x and t. It would be nice to have a simpler relation.

Notice that in the case $\alpha = 1$ the a_j coefficients vanish and we get

$$-\beta \sqrt{\frac{\rho_0}{\epsilon}}\, \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\, \frac{\pi}{L} t\right) \cos\left(\frac{\pi}{L} x\right) < 1$$

as the general condition which clearly gives $\beta < 1$.

Let us assume for a moment that if a negative value for the derivative w_x appears at some time, that negative value will appear for some time at the right endpoint $x = L$. This at least simplifies the condition to

$$\beta\sqrt{\frac{\rho_0}{\epsilon}}\ \sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{\pi}{L}t\right)$$
$$< 1 - \frac{4(\alpha-1)}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{(2k+1)\pi}{L}t\right)$$

for all t. At least in the special case $\epsilon/\rho_0 = 1 = L$, the Fourier series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{(2k+1)\pi}{L}t\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left((2k+1)\pi t\right)$$

is a square wave of amplitude $\pi/4$ and period 2. Thus we are led to the condition

$$\beta < 1 - (\alpha - 1) = 2 - \alpha. \tag{9}$$

This does have the advantage of giving $\beta < 1$ when $\alpha = 1$ and $\beta = 0$ when α tends to 2. Checking the solution when $\alpha = 3/2$ and using about 100 terms in the Fourier expansion suggests this is the correct value, so (9) is probably the simplest condition suggested by the animations.

(ii) Notice that when $\beta = 0$ the condition (9) certainly holds.

(iii) If we take $\beta = 0$ then we have

$$w(x,t) = x + \frac{4L(\alpha-1)}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{(2k+1)\pi}{L}t\right)\sin\left(\frac{(2k+1)\pi}{L}x\right)$$

and

$$w_t = -\frac{4(\alpha-1)}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\sin\left(\sqrt{\frac{\epsilon}{\rho_0}}\ \frac{(2k+1)\pi}{L}t\right)\sin\left(\frac{(2k+1)\pi}{L}x\right).$$

Notice that when

$$t_m = mL\sqrt{\frac{\rho_0}{\epsilon}} \qquad \text{for } m = 0, 1, 2, 3, \ldots$$

then $w_t(x, t_m) \equiv 0$. You can see these as the maximum right and left compressions in the animations. If $\beta > 0$, then the situation is more complicated. The first x mode gets a phase shift, and it's not entirely clear that such "rest points" exist. Clearly when $\alpha = 1$ there is no coupling in the first mode and rest points exist. However, when there is an initial "tent" displacement coupled with an initial sinusoidal velocity, there appears to be a kind of "traveling wave" that circulates from left to right and back as the entire medium oscillates through maximum compressions. These two should be out of phase in general creating internal motion at all times. This is suggested by the animation with $\alpha = 1.3$ and $\beta = 1.5$.

Problem 8 (energy of internal oscillations) Let w be a solution of (8).

(a) Explain why

$$T(t) = \int_0^L \left(\frac{\rho_0}{2} w_t^2 + \frac{\epsilon}{2} (w_x - 1)^2 \right) dx$$

is a reasonable quantity to model the total energy of the system.

(b) Show $T(t)$ is constant with value $T(0)$.

(c) Can you make a comparison between the frequency of solutions for (7) and (8)?

Generalized Solutions of the Wave Equation

A function $w \in C^2((0,L) \times (0,\infty))$ is a **classical solution** of the wave equation $\rho_0 w_{tt} = \epsilon w_{xx}$ if one can substitute w into the PDE, evaluating the derivatives in the usual fashion, and verify that the PDE holds. In Problem 7 you were asked to solve an initial/boundary value problem involving this equation. In the following problem you are asked to return to the solution you found and contemplate whether or not what you have found is a classical solution.

Problem 9 (generalized solutions of the wave equation) Consider again the initial boundary value problem (8) with initial spatial deformation

$$w_0(x) = \begin{cases} \alpha x, & 0 < x < L/2 \\ (\alpha - 1)L + (2 - \alpha)x, & L/2 < x < L \end{cases}$$

and initial velocity $v_0(x) \equiv 0$.

(a) Calculate

$$\frac{d^2}{dx^2} w_0(x).$$

What does this suggest about the initial acceleration $w_{tt}(x, 0)$?

(b) Use d'Alembert's formula/the method of characteristics to find $w(x,t)$ in the triangular domain

$$A = \left\{ (x,t) : 0 < t < \frac{L}{2} - \left| x - \frac{L}{2} \right|, 0 < x < L \right\}.$$

(c) Is the solution you found in part **(b)** consistent with the solution you found in Problem 7?

(d) Calculate

$$\frac{\partial^2 w}{\partial x^2}(x,t) \qquad \text{for} \qquad (x,t) \in A.$$

What does this suggest about the solution you found in Problem 7?

(e) Can you give some justification for calling the function $w = w(x,t)$ you have found a "solution" of (8)? From whence comes the acceleration?

Solution:

(a)

$$\frac{d^2}{dx^2} w_0(x) = 0$$

at all points except $x = L/2$. This suggest classically that $w_{tt}(x,0) \equiv 0$ at almost all points, so one might expect no acceleration and hence no motion. Note however this is far from what the Fourier series expansion suggests.

(b) From d'Alembert's formula for u with initial wave form $w_0(x)$ and zero initial velocity v_0 is given by the average

$$w(x,t) = \frac{1}{2}[w_0(x-t) + w_0(x+t)].$$

For (x,t) in the triangular domain A and $0 < x < L/2 - t$ we have $0 < x - t < x + t < L/2$. Thus $w_0(x-t) = \alpha(x-t)$ and $w_0(x+t) = \alpha(x+t)$. Therefore,

$$w(x,t) = \alpha x \qquad \text{for} \qquad 0 < x < L/2 - t.$$

When $L/2 - t < x < L/2 + t$, then $x - t < L/2 < x + t$, so $w_0(x-t) = \alpha(x-t)$ and $w_0(x+t) = (\alpha - 1)L + (2 - \alpha)(x+t)$. This means

$$w_0(x-t) + w_0(x+t) = (\alpha-1)L + 2(x+t) - 2\alpha t = (\alpha-1)L + 2x - 2(\alpha-1)t \qquad \text{for} \qquad L/2 - t < x <$$

so d'Alembert's formula gives

$$w(x,t) = x + \frac{\alpha - 1}{2} L - (\alpha - 1)t \qquad \text{for} \qquad L/2 - t < x < L/2 + t.$$

Finally, when $L/2 + t < x < L$, then $L/2 < x - t < x + t < L$ so $w_0(x - t) = (\alpha - 1)L + (2 - \alpha)(x - t)$ and $w_0(x - t) = (\alpha - 1)L + (2 - \alpha)(x + t)$, so

$$w(x,t) = (\alpha - 1)L + (2 - \alpha)x \qquad \text{for} \qquad L/2 + t < x < L.$$

Putting these together we see d'Alembert's formula gives a piecewise affine function

$$w(x,t) = \begin{cases} \alpha x, & 0 < x < L/2 - t \\ x + \frac{\alpha - 1}{2} L - (\alpha - 1)t, & L/2 - t < x < L/2 + t \\ (\alpha - 1)L + (2 - \alpha)x, & L/2 + t < x < L. \end{cases}$$

(c) This appears to be precisely the same solution obtained using Fourier series expansion when restricted to the triangular domain A.

(d) There are two singular lines Σ_- and Σ_+ separating the triangular domain A into three regions as indicated in Figure 2 The function $w = w(x,t)$ is continuous

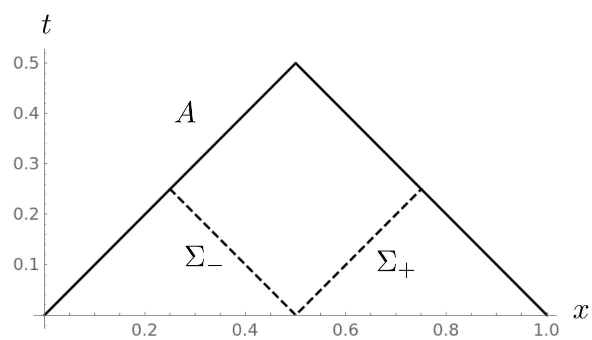

Figure 2: The triangular domain A with singular curves.

on A and differentiable on $A \backslash (\Sigma_- \cup \Sigma_+)$ with the second partial

$$\frac{\partial^2 w}{\partial x^2}(x,t)$$

vanishing for $(x,t) \in A \backslash (\Sigma_- \cup \Sigma_+)$. This second partial is not defined by $(x,t) \in \Sigma_- \cup \Sigma_+$.

While the functions $w = w(x,t)$ found using Fourier series and d'Alembert's formula agree with one another, they are not (classical) solutions of the wave equation.

(e) Somehow acceleration is concentrated at the "corners" along Σ_- and Σ_+ where

$$\frac{\partial^2 w}{\partial x^2}(x,t)$$

has a jump discontinuity. We can say w has weak first partial derivatives with respect to x and t. Denoting these functions by

$$W_t = \begin{cases} 0, & 0 < x < L/2 - t \\ (\alpha - 1), & L/2 - t < x < L/2 + t \\ 0, & L/2 + t < x < L \end{cases}$$

and

$$W_x = \begin{cases} x, & 0 < x < L/2 - t \\ 1, & L/2 - t < x < L/2 + t \\ 2 - \alpha, & L/2 + t < x < L, \end{cases}$$

we have

$$\int_A W_x \phi = -\int_A w \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad \int_A W_t \phi = -\int_A w \frac{\partial \phi}{\partial t}$$

for every $\phi \in C_c^\infty(A)$. These functions are determined uniquely except on the singular sets Σ_- and Σ_+ where the value(s) cannot effect such integrals anyway. That is, they are determined uniquely in $L^1(A)$.

These still do not get us to any kind of notion of solution for the wave equation however. The functions W_x and W_t do not have weak derivatives.

One needs to associate with the weak derivatives W_x and W_t integral functionals. In fact, the functionals given above are the desired integral functionals:

$$\mathcal{F}_t[\phi] = -\int_A w \frac{\partial \phi}{\partial t} \qquad \text{and} \qquad \mathcal{F}_x[\phi] = -\int_A w \frac{\partial \phi}{\partial x}.$$

Then one can attempt to define distributional derivatives

$$\mathcal{D}_t \mathcal{F}_t : C_c^\infty(A) \to \mathbb{R} \qquad \text{and} \qquad \mathcal{D}_x \mathcal{F}_x : C_c^\infty(A) \to \mathbb{R}$$

of $\mathcal{F}_t$ and $\mathcal{F}_x$ (and formulate a version of our initial/boundary value problem for the wave equation in terms of distributions. If one does this correctly, the d'Alembert solution (which is the same as the Fourier series solution) can be said to solve the wave equation in the sense of distributions.

Bessel Functions

Bessel's ODE is

$$x^2\ y'' + x\ y' + (x^2 - k^2)y = 0. \tag{10}$$

This is an equation for the function $y = y(x)$. Notice there is a parameter k in the equation, so we do not really have just one equation here but rather a family of ODEs having somewhat comparable forms. In fact, the ODE in (10) is called **Bessel's equation of order** k. Let us call $B = B_k : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by

$$B[y] = x^2\ y'' + x\ y' + (x^2 - k^2)y$$

the **Bessel operator**.

Like $y'' + \omega^2 y = 0$ has a standard basis of solutions $\{\cos(\omega x), \sin(\omega x)\}$ and $y'' - \omega^2 y = 0$ has a (standard) basis of solutions $\{\cosh(\omega x), \sinh(\omega x)\}$, Bessel's ODE has a standard basis of solutions $\{J_k(x), Y_k(x)\}$. The following exercise gives you the opportunity to learn some things about these new special functions.

Problem 10 (Bessel functions; Haberman section 7.7-8)

(a) Is the Bessel operator linear?

(b) Explain why Bessel's equation is not an equidimensional/Euler equation as encountered when using polar coordinates to solve Laplace's equation on a disk.

(c) Is

$$\begin{cases} B[y] + \lambda y = 0, & 0 < x < L \\ y(0) = 0 = y(L) \end{cases}$$

a regular Sturm-Liouville problem? (Explain why or why not.)

(d) Find a power series solution J_0 of Bessel's equation of order zero with initial values

$$J_0(0) = 1$$
$$J_0'(0) = 0.$$

(e) For $k = 0$ the second standard basis solution is given by

$$Y_0(x) = \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right) J_0(x) + u_2(x) \tag{11}$$

where $\gamma \doteq -.5772157$ is Euler's constant and u_2 is given by a power series (which is regular C^∞ at $x = 0$ in particular with $u_2(0) = 0$). Show

$$u_1(x) = J_0(x)\ln x$$

is a solution of the ODE

$$B_0[u] = 2x\ J_0'(x).$$

Note: The point of part **(e)** is that one can get a linearly independent solution for the zero order Bessel equation $B[y] = 0$ in addition to the power series solution J_0 from part **(d)** by finding a power series solution of $B_0[u_2] = -2x\ J_0'(x)$. This is fairly easy since J_0 is already given as a power series; you just incorporate the coefficients. Then $\{J_0, u_1 + u_2\} = \{J_0, \ln x\ J_0 + u_2\}$ is a basis of solutions for the zero order Bessel ODE. You know these two solutions are linearly independent because the first one J_0 is bounded at $x = 0$ with $J_0(0) = 1$ and the second one $u_1 + u_2$ is unbounded at/near $x = 0^+$. The specific solution Y_0 given in (11) above is obtained by the same method; just find the linear inhomogeneous ODE $L[u_1] = f(x)$ satisfied by

$$u_1 = \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right) J_0(x)$$

and solve the IVP

$$\begin{cases} L[u_2] = -f(x), & x \geq 0 \\ u_2(0) = 0 = u_2'(0). \end{cases}$$

The explanation for the specific normalizations/constants etc. used in Y_0 is more complicated, but I hope you can see the basic method. For the next step/more details look up (and learn) the "Method of Froebenius". A good reference is *Advanced Engineeering Mathematics* by Peter V. O'Neil.

Solution:

(a) The Bessel operator is linear because

$$B[y_1 + y + 2] = B[y_1] + B[y_2] \qquad \text{and} \qquad B[cy] = cB[y]$$

for any $y, y_1, y_2 \in C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$.

(b) Bessel's equation is not an equidimensional/Euler equation because the zero order term

$$(x^2 - k^2)y$$

is not a constant times y. The equation

$$x^2 y + xy' - k^2 y = 0$$

is equidimensional and can be solved with a substitution $y = x^\alpha$, but the $x^2 y$ term results in the inability to get a polynomial equation for α when such a substitution is attempted.

(c) The two point boundary value problem

$$\begin{cases} B[y] + \lambda y = 0, \quad 0 < x < L \\ y(0) = 0 = y(L) \end{cases}$$

is not a regular Sturm-Liouville problem. The equation is

$$(xy')' + xy - \frac{k^2}{x} y = 0$$

is of Sturm-Liouville form with coefficients $p(x) = x$, $q(x) = 0$, and $\mu(x) = -k^2/x$. The leading coefficent function $p(x) = x$ vanishes at $x = 0$ which is a singular condition for a Sturm-Liouville problem. Also, when $k \neq 0$, the weight function $\mu(x) = -k^2/x$ is not continuous at $x = 0$.

(d) Putting

$$y(x) = \sum_{j=0}^{\infty} a_j x^j$$

and $k = 0$ we want

$$\sum_{j=2}^{\infty} j(j-1)a_j x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^{j+2} = 0$$

along with $a_0 = 1$ and $a_1 = 0$. The coefficient of x (on the left) is $a_1 = 0$, so that is okay. Shifting indices in the last sum we can rewrite the series condition as

$$\sum_{j=2}^{\infty} (j^2 a_j + a_{j-2}) x^j = 0.$$

From this we get the recursion relation

$$a_j = -\frac{a_{j-2}}{j^2} \qquad \text{for } j = 2, 3, 4, \ldots.$$

In particular, $a_2 = -a_0/4 = -1/4$ and $a_3 = -a_1/9 = 0$. It follows more generally that

$$a_{2k} = \frac{(-1)^k}{(2k)^2(2k-2)^2 \cdots 2^2} = \frac{(-1)^k}{2^{2k}\ (k!)^2}, \qquad k = 1, 2, 3, \ldots$$

and $a_{2k+1} = 0$ for $k = 0, 1, 2, 3, \ldots$. Thus,

$$J_0(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}\ (k!)^2} x^{2k}.$$

(e) Setting

$$u_1(x) = J_0(x) \ln x,$$

we have

$$u_1' = \frac{1}{x} J_0 + J_0' \ln x \qquad \text{and} \qquad u_1'' = -\frac{1}{x^2} J_0 + \frac{2}{x} J_0' + J_0'' \ln x.$$

Therefore,

$$\begin{aligned} B_0[u_1] &= -J_0 + 2xJ_0' + x^2 J_0'' \ln x + J_0 + xJ_0 \ln x + x^2 J_0 \ln x \\ &= 2x\ J_0'(x) + B[J_0] \ln x \\ &= 2x\ J_0'(x). \end{aligned}$$

Thus, $u_1(x) = J_0 \ln x$ is one solution of the inhomogeneous ODE

$$x^2 u'' + xu' + x^2 y = 2xJ_0'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k+1}\ (k!)^2} x^{2k+1}. \tag{12}$$

This particular solution is unbounded at $x = 0$. Thus, if we can find another particular solution $u = u_2$ of (12) with for example $u_2(0) = 0 = u_2'(0)$, then

$$\{J_0, u_1 - u_2\}$$

will be a basis of solutions for $B[y] = 0$. This is basically how one constructs the zero order Bessel function of the second kind.

(These ideas also extend to Bessel functions of higher order.)