Assignment 9: Wave Equation (review) ***under construction*** ***Posted Friday March 27, 2026*** (more problems coming later) Due Wednesday April 22, 2026

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For the problems below L denotes a positive real number. By the **wave equation** we mean the second order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

for a real valued function $u \in C^2(\Omega \times [0, T))$ of n spatial variables and one time variable, that is the PDE $u_{tt} = \Delta u$.

Problems 1-3 are precisely the same as Problems 8-10 of Assignment 8. You do not need to do them again for this assignment. You can just copy your answers from what you turned in on Assignment 8, but you may want to review these problems as Problem 4 is the final problem in this series of problems. Also, it may not hurt you to do them (again).

Problem 1 (factoring the wave operator; same as Problem 8 of Assignment 8) The **wave operator** is also called the d'Alembertian, and it is denoted by $\Box : C^2(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ with

$$\Box u = u_{tt} - u_{xx}.$$

Like the Laplacian $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $\Delta u = u_{xx}$ and the heat operator $L : C^2(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ by $Lu = u_t - u_{xx}$, the d'Alembertian is a second order partial differential operator.

Let $T : C^1(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ be the first order transport operator given by

$$Tu = u_t - u_x.$$

(a) Find a first order linear operator $S : C^1(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ for which

$$\Box u = S \circ T.$$

(b) Compute $T \circ S$.

(c) An operator $A : V \to W$ defined on a vector space V and taking values in a vector space W is **linear** if

$$A(cu) = cAu \qquad \text{for every } c \in \mathbb{R} \text{ and } u \in V$$

and

$$A(u + v) = Au + Av \qquad \text{for every } u, v \in V.$$

(i) Show the wave operator is linear on $C^2(\mathbb{R} \times (0, \infty))$.

(ii) Show the operator T is linear on $C^1(\mathbb{R} \times (0, \infty))$.

(iii) Show the Laplace operator Δ is linear on $C^2((a, b))$ where $a, b \in \mathbb{R}$ with $a < b$.

(iv) Show the heat operator L is linear on $C^2((a, b) \times (0, \infty))$ where $a, b \in \mathbb{R}$ with $a < b$.

Problem 2 (The wave equation on all of $\mathbb{R}$; same as Problem 9 of Assignment 8) Consider the initial value problem (IVP) for the wave equation:

$$\begin{cases} u_{tt} = u_{xx}, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ u_t(x, 0) = 0, & x \in \mathbb{R} \end{cases} \tag{1}$$

where u_0 is a given function with $u_0 \in C^2(\mathbb{R})$. Assuming $u \in C^2(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ is a solution of (1) and $w = Tu \in C^1(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ where S and T are the factor operators from Problem 1, find an appropriate initial value problem for a transport equation satisfied by w.

In particular, find an appropriate initial value w_0 for the problem

$$\begin{cases} Sw = 0, & \text{on } \mathbb{R} \times (0, \infty) \\ w(x, 0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Problem 3 (first order linear equation; same as Problem 10 of Assignment 8) Consider the problem (2) with the initial condition you found in Problem 2. Solve that problem by completing the following steps:

(a) Consider a parameterized path $\gamma : [0, \infty) \to \mathbb{R} \times [0, \infty)$ given by $\gamma(t) = (\xi(t), t)$ for some real valued spatial function $x = \xi(t)$. Use the chain rule to compute

$$\frac{d}{dt} w \circ \gamma(t). \tag{3}$$

(b) Compare your result from (3) to the PDE from (2). Given any initial starting point $x_0 \in \mathbb{R}$ find an appropriate ODE for $\xi : [0, \infty) \to \mathbb{R}$ based on your comparison, and solve the ODE with the initial condition $\xi(0) = x_0$.

(c) With your solution for ξ from part **(b)** which should depend on x_0, consider for an arbitrary point $(x, t) \in \mathbb{R} \times (0, \infty)$ the equation

$$\gamma(t) = (x, t). \tag{4}$$

Choose x_0 so that (4) is satisfied.

(d) If you made the correct choice of ODE in part **(b)** you should now know the value of the quantity in (3), which should tell you

$$w(\xi(t), t) = w_0(x_0).$$

If you made the correct choice of w_0 in Problem 2 you should now know the solution of the problem (2) in terms of u_0 from the IVP in Problem 2.

Problem 4 (another first order linear equation) Consider the inhomogeneous (forced) initial value problem for $u \in C^1(\mathbb{R} \times [0, \infty))$

$$\begin{cases} u_t - u_x = w, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{5}$$

where $u_0, w \in C^0(\mathbb{R})$ are given initial and spatial forcing functions.

(a) Consider a propagating curve $\gamma(t) = (\xi(t), t)$ with $\xi(0) = x_0 \in \mathbb{R}$. Calculate

$$\frac{d}{dt} u \circ \gamma(t) \tag{6}$$

and pose an appropriate ODE for ξ based on comparison with the operator $Tu = u_t - u_x$ in the PDE.

(b) Draw a picture of the curve parameterized by γ in $\mathbb{R} \times [0, \infty)$.

(c) How would you characterize the propagation of x_0 induced by γ? (Give speed and direction.)

(d) Derive an ODE for $u \circ \gamma$ based on your work above and computation of the derivative in (6).

(e) Couple your ODE from part **(d)** with an appropriate initial condition to find a formula for $u \circ \gamma(t)$ as a function of x_0. Be careful with the argument of w.

(f) Solve the equation $\gamma(t) = (x, t)$ for the starting point x_0.

(g) Substitute (x, t) in for $\gamma(t)$ along with the value you found for x_0 in part **(f)** into the formula you found in part **(e)** to solve the problem (5).

Problem 5 (Problem 4 above; finite propagation speed of a signal) Take the specific choice(s)

$$u_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad w \equiv 0$$

in the formula you obtained in part **(g)** of Problem 4.

(a) Plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(b) If you think of the behavior of u_0 on its support as a "signal," how long does it take for information from the signal to be communicated at $x = -10$? At what time has the signal passed $x = -10$?