Assignment 9: Wave Equation (review)
Due Wednesday April 22, 2026

John McCuan

For the problems below L denotes a positive real number. By the **wave equation** we mean the second order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

for a real valued function $u \in C^2(\Omega \times [0, T))$ of n spatial variables and one time variable, that is the PDE $u_{tt} = \Delta u$.

Problems 1-3 are precisely the same as Problems 8-10 of Assignment 8. You do not need to do them again for this assignment. You can just copy your answers from what you turned in on Assignment 8, but you may want to review these problems as Problem 4 is the final problem in this series of problems. Also, it may not hurt you to do them (again).

Problem 1 (factoring the wave operator; same as Problem 8 of Assignment 8) The **wave operator** is also called the d'Alembertian, and it is denoted by $\Box : C^2(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ with

$$\Box u = u_{tt} - u_{xx}.$$

Like the Laplacian $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $\Delta u = u_{xx}$ and the heat operator $L : C^2(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ by $Lu = u_t - u_{xx}$, the d'Alembertian is a second order partial differential operator.

Let $T : C^1(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ be the first order transport operator given by

$$Tu = u_t - u_x.$$

(a) Find a first order linear operator $S : C^1(\mathbb{R} \times (0, \infty)) \to C^0(\mathbb{R} \times (0, \infty))$ for which

$$\Box u = S \circ T.$$

(b) Compute $T \circ S$.

(c) An operator $A : V \to W$ defined on a vector space V and taking values in a vector space W is **linear** if

$$A(cu) = cAu \qquad \text{for every } c \in \mathbb{R} \text{ and } u \in V$$

and

$$A(u + v) = Au + Av \qquad \text{for every } u, v \in V.$$

(i) Show the wave operator is linear on $C^2(\mathbb{R} \times (0, \infty))$.

(ii) Show the operator T is linear on $C^1(\mathbb{R} \times (0, \infty))$.

(iii) Show the Laplace operator Δ is linear on $C^2((a, b))$ where $a, b \in \mathbb{R}$ with $a < b$.

(iv) Show the heat operator L is linear on $C^2((a, b) \times (0, \infty))$ where $a, b \in \mathbb{R}$ with $a < b$.

Problem 2 (The wave equation on all of $\mathbb{R}$; same as Problem 9 of Assignment 8) Consider the initial value problem (IVP) for the wave equation:

$$\begin{cases} u_{tt} = u_{xx}, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases} \tag{1}$$

where u_0 is a given function with $u_0 \in C^2(\mathbb{R})$. Assuming $u \in C^2(\mathbb{R} \times (0,\infty)) \cap C^0(\mathbb{R} \times [0,\infty))$ is a solution of (1) and $w = Tu \in C^1(\mathbb{R} \times (0,\infty)) \cap C^0(\mathbb{R} \times [0,\infty))$ where S and T are the factor operators from Problem 1, find an appropriate initial value problem for a transport equation satisfied by w.

In particular, find an appropriate initial value w_0 for the problem

$$\begin{cases} Sw = 0, & \text{on } \mathbb{R} \times (0, \infty) \\ w(x,0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Problem 3 (first order linear equation; same as Problem 10 of Assignment 8) Consider the problem (2) with the initial condition you found in Problem 2. Solve that problem by completing the following steps:

(a) Consider a parameterized path $\gamma : [0,\infty) \to \mathbb{R} \times [0,\infty)$ given by $\gamma(t) = (\xi(t), t)$ for some real valued spatial function $x = \xi(t)$. Use the chain rule to compute

$$\frac{d}{dt} w \circ \gamma(t). \tag{3}$$

(b) Compare your result from (3) to the PDE from (2). Given any initial starting point $x_0 \in \mathbb{R}$ find an appropriate ODE for $\xi : [0,\infty) \to \mathbb{R}$ based on your comparison, and solve the ODE with the initial condition $\xi(0) = x_0$.

(c) With your solution for ξ from part **(b)** which should depend on x_0, consider for an arbitrary point $(x,t) \in \mathbb{R} \times (0,\infty)$ the equation

$$\gamma(t) = (x,t). \tag{4}$$

Choose x_0 so that (4) is satisfied.

(d) If you made the correct choice of ODE in part **(b)** you should now know the value of the quantity in (3), which should tell you

$$w(\xi(t), t) = w_0(x_0).$$

If you made the correct choice of w_0 in Problem 2 you should now know the solution of the problem (2) in terms of u_0 from the IVP in Problem 2.

Problem 4 (another first order linear equation) Consider the inhomogeneous (forced) initial value problem for $u \in C^1(\mathbb{R} \times [0, \infty))$

$$\begin{cases} u_t - u_x = w, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{5}$$

where $u_0, w \in C^0(\mathbb{R})$ are given initial and spatial forcing functions.

(a) Consider a propagating curve $\gamma(t) = (\xi(t), t)$ with $\xi(0) = x_0 \in \mathbb{R}$. Calculate

$$\frac{d}{dt} u \circ \gamma(t) \tag{6}$$

and pose an appropriate ODE for ξ based on comparison with the operator $Tu = u_t - u_x$ in the PDE.

(b) Draw a picture of the curve parameterized by γ in $\mathbb{R} \times [0, \infty)$.

(c) How would you characterize the propagation of x_0 induced by γ? (Give speed and direction.)

(d) Derive an ODE for $u \circ \gamma$ based on your work above and computation of the derivative in (6).

(e) Couple your ODE from part **(d)** with an appropriate initial condition to find a formula for $u \circ \gamma(t)$ as a function of x_0. Be careful with the argument of w.

(f) Solve the equation $\gamma(t) = (x, t)$ for the starting point x_0.

(g) Substitute (x, t) in for $\gamma(t)$ along with the value you found for x_0 in part **(f)** into the formula you found in part **(e)** to solve the problem (5).

Problem 5 (Problem 4 above; finite propagation speed of a signal) Take the specific choice(s)

$$u_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad w \equiv 0$$

in the formula you obtained in part **(g)** of Problem 4.

(a) Plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(b) If you think of the behavior of u_0 on its support as a "signal," how long does it take for information from the signal to be communicated at $x = -10$? At what time has the signal passed $x = -10$?

Problems 6-8: Oscillators

Problems 6-8 suggest a comparison between the modeling of a mass on a spring using ODEs and the modeling of an internally deformed spring (or more properly thin/one-dimensional elastic medium) with internal mass using the wave equation.

Recall the modeling of a harmonic oscillator (mass on a spring) associated with the IVP

$$\begin{cases} m\ddot{x} = -k(x - L), & t \in \mathbb{R} \\ x(0) = x_0, \\ \dot{x}(0) = v_0. \end{cases} \tag{7}$$

Given an initial position x_0 and initial velocity v_0 for the mass m on a (massless) spring with Hooke's constant k, a nonnegative number modeling the **total energy of the initial configuration** is given by

$$T_0 = \frac{1}{2}mv_0^2 + \frac{1}{2}k(x_0 - L)^2.$$

Problem 6 (energy)

(a) Draw a picture of the physical system modelted by (7).

(b) Evaluate the integral

$$\int_0^x k(\xi - L)\,d\xi$$

and explain why this value is used to model the potential energy for a harmonic oscillator. Hint: potential energy equals "force times distance."

(c) Let $T = T(t)$ denote the sum of the kinetic energy $m\dot{x}^2/2$ and the potential energy associated at each time t with a solution of (7). Show T is (the) constant T_0.

(d) How would this discussion change if (linear) damping were incorporated in the model?

Problem 7 (internal oscillations) Assume an elastic medium with elasticity ϵ and constant lineal density (at equilibrium) ρ_0 is modeled using the initial/boundary value problem

$$\begin{cases} \rho_0 w_{tt} = \epsilon w_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ w(x,0) = w_0(x), & x \in (0,L) \\ w_t(x,0) = v_0(x), & x \in (0,L) \\ w(0,t) = 0 = w(L,t), & t > 0. \end{cases} \tag{8}$$

(a) Solve this problem using separation of variables and Fourier series. Hint(s): Set $u = w - x$ and solve the problem for u with homogeneous boundary values. Your answer should involve coefficients depending on the functions w_0 and v_0.

(b) Let α and β be numbers with $1 < \alpha < 2L$ and $\beta \geq 0$. Take

$$w_0(x) = \begin{cases} \alpha x, & 0 < x < L/2 \\ (\alpha - 1)L + (2 - \alpha)x, & L/2 < x < L \end{cases} \qquad \text{and} \qquad v_0(x) = \beta \sin x$$

and evaluate the coefficients you found in part **(a)**.

(c) Animate the internal oscillations modeled here (with time as the animation parameter) for specific numerical values of L, ρ_0, ϵ, α and β.

(i) What relations must be maintained among the parameters in order to preserve the modeling assumption $w_x(x,t) > 0$ for all $(x,t) \in (0,L) \times (0,\infty)$?

(ii) If $\beta = 0$, do the relations you found in part **(a)** hold?

(iii) Do you see specific (positive) times t for which

$$w_t(x,t) \equiv 0, \qquad 0 < x < L \ ?$$

What is the frequency associated with these times? (You may wish to restrict attention to $\beta = 0$ here.)

Problem 8 (energy of internal oscillations) Let w be a solution of (8).

(a) Explain why

$$T(t) = \int_0^L \left(\frac{\rho_0}{2} w_t^2 + \frac{\epsilon}{2}(w_x - 1)^2 \right) dx$$

is a reasonable quantity to model the total energy of the system.

(b) Show $T(t)$ is constant with value $T(0)$.

(c) Can you make a comparison between the frequency of solutions for (7) and (8)?

Generalized Solutions of the Wave Equation

A function $w \in C^2((0, L) \times (0, \infty))$ is a **classical solution** of the wave equation $\rho_0 w_{tt} = \epsilon w_{xx}$ if one can substitute w into the PDE, evaluating the derivatives in the usual fashion, and verify that the PDE holds. In Problem 7 you were asked to solve an initial/boundary value problem involving this equation. In the following problem you are asked to return to the solution you found and contemplate whether or not what you have found is a classical solution.

Problem 9 (generalized solutions of the wave equation) Consider again the initial boundary value problem (8) with initial spatial deformation

$$w_0(x) = \begin{cases} \alpha x, & 0 < x < L/2 \\ (\alpha - 1)L + (2 - \alpha)x, & L/2 < x < L \end{cases}$$

and initial velocity $v_0(x) \equiv 0$.

(a) Calculate

$$\frac{d^2}{dx^2} w_0(x).$$

What does this suggest about the initial acceleration $w_{tt}(x, 0)$?

(b) Use d'Alembert's formula/the method of characteristics to find $w(x, t)$ in the triangular domain

$$A = \left\{ (x, t) : 0 < t < \frac{L}{2} - \left| x - \frac{L}{2} \right|, 0 < x < L \right\}.$$

(c) Is the solution you found in part **(b)** consistent with the solution you found in Problem 7?

(d) Calculate

$$\frac{\partial^2 w}{\partial x^2}(x, t) \qquad \text{for} \qquad (x, t) \in A.$$

What does this suggest about the solution you found in Problem 7?

(e) Can you give some justification for calling the function $w = w(x, t)$ you have found a "solution" of (8)? From whence comes the acceleration?

Bessel Functions

Bessel's ODE is

$$x^2\ y'' + x\ y' + (x^2 - k^2)y = 0. \tag{9}$$

This is an equation for the function $y = y(x)$. Notice there is a parameter k in the equation, so we do not really have just one equation here but rather a family of ODEs having somewhat comparable forms. In fact, the ODE in (9) is called **Bessel's equation of order** k. Let us call $B = B_k : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by

$$B[y] = x^2\ y'' + x\ y' + (x^2 - k^2)y$$

the **Bessel operator**.

Like $y'' + \omega^2 y = 0$ has a standard basis of solutions $\{\cos(\omega x), \sin(\omega x)\}$ and $y'' - \omega^2 y = 0$ has a (standard) basis of solutions $\{\cosh(\omega x), \sinh(\omega x)\}$, Bessel's ODE has a standard basis of solutions $\{J_k(x), Y_k(x)\}$. The following exercise gives you the opportunity to learn some things about these new special functions.

Problem 10 (Bessel functions; Haberman section 7.7-8)

(a) Is the Bessel operator linear?

(b) Explain why Bessel's equation is not an equidimensional/Euler equation as encountered when using polar coordinates to solve Laplace's equation on a disk.

(c) Is

$$\begin{cases} B[y] + \lambda y = 0, & 0 < x < L \\ y(0) = 0 = y(L) \end{cases}$$

a regular Sturm-Liouville problem? (Explain why or why not.)

(d) Find a power series solution J_0 of Bessel's equation of order zero with initial values

$$J_0(0) = 1$$
$$J_0'(0) = 0.$$

(e) For $k = 0$ the second standard basis solution is given by

$$Y_0(x) = \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right) J_0(x)$$

where $\gamma \doteq -.5772157$ is Euler's constant. Show

$$y(x) = J_0(x)\ln x$$

is a solution of the zero order Bessel equation.