Assignment 8: Wave Equation
Due Wednesday April 8, 2026

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For the problems below L denotes a positive real number. By the **wave equation** we mean the second order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

for a real valued function $u \in C^2(\Omega \times [0,T))$ of n spatial variables and one time variable, that is the PDE $u_{tt} = \Delta u$.

Problem 1 (calculus of variations) Consider the length functional $\ell : \mathcal{A} \to [0,\infty)$ with values

$$\ell[f] = \int_0^L \sqrt{1 + [f'(x)]^2}\, dx$$

where

$$\mathcal{A} = \left\{ f \in C^1[0,L] : f(0) = 0,\ f(L) = c \right\}.$$

(a) Let $\phi \in C_c^\infty(0,L)$ be fixed and compute

$$\delta \ell_f[\phi] = \frac{d}{d\epsilon} \ell[f + \epsilon\phi]\Big|_{\epsilon=0}.$$

(b) Assuming f is a minimizer of ℓ, that is $\ell[f] \le \ell[g]$ for all $g \in \mathcal{A}$, and $\phi \in C_c^\infty(0,L)$ is fixed, draw the graph of the function $h : \mathbb{R} \to \mathbb{R}$ with values

$$h(\epsilon) = \ell[f + \epsilon\phi].$$

Be sure to label the value $h(0)$ and indicate the value $h'(0)$.

(c) Use integration by parts and the fundamental lemma of the calculus of variations to find the Euler-Lagrange ODE satisfied by a minimizer $f \in \mathcal{A}$ of the length functional ℓ.

Problem 2 (wave equation; Fourier series) Find a Fourier sine series solution for the initial/boundary value problem

$$\begin{cases} u_{tt} = \kappa u_{xx} & \text{on } (0, L) \times (0, \infty) \\ u(0, t) = 0 = u(L, t), & t > 0 \\ u(x, 0) = x(L - x) = u_t(x, 0), & 0 < x < 1. \end{cases}$$

for the wave equation.

1-D Wave Equation— Derivation

The wave equation provides a model for relatively large scale (gross) internal physical **motion** of a medium due to internal elastic forces in a manner somewhat analogous to the provision of a model for the (small scale/mollecular) **diffusion** of an (externally) imposed "condition" like the change in excitement of molecules in place (by the transfer of thermal energy) or the change in concentration due to microscipic migration. Recall that at the heart of the derivation of the heat equation lies the rate of change of a "total" or "accumulated" quantity in a subregion:

$$\frac{d}{dt} \int_R \Theta = \frac{d}{dt} \int_R cu.$$

This quantity is then related to a flux integral. The wave equation requires a very different interpretation and a fundamentally new principle much more closely related to Newton's law

$$f = ma$$

for the motion of a "particle" with mass m in terms of a force. One seeks here an expression for the **acceleration** on a subregion deformed or "moved out of place" from some reference position. We consider first the notion of deformation.

Problem 3 (homogeneous 1-D deformations) Let $I_0 = [0, L]$ model a deformable one-dimensional medium within a line (modeled by) $\mathbb{R}$. Accordingly we associate with each $x \in [0, L]$ and each positive time t a location of deformation $w = w(x, t)$. Physically, the function $w : [0, L] \times [0, T) \to \mathbb{R}$ should be restricted in various ways. For example, a natural restriction is that w should be spatially **one-to-one** for each fixed time t, i.e., no two distinct physical locations $x_1 \neq x_2$ (presumably modeling positions of positive mass density) should be assigned to the same deformed locations $w(x_1, t)$ and $w(x_2, t)$ at the same time. A simple family of time independent deformations having this property are given by

$$w = h(x) = cx$$

where $c > 0$ is constant.

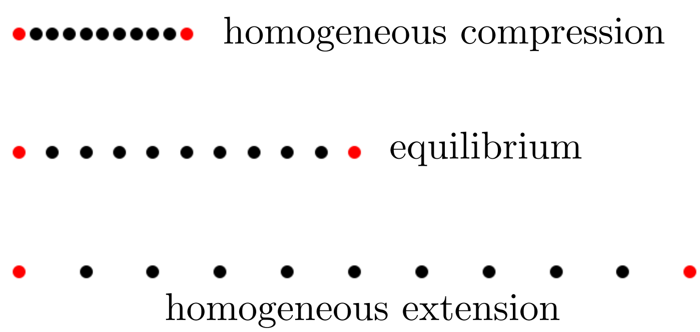

Figure 1: homogeneous 1-D deformations

(a) If ρ_0 is a constant equilibrium mass density associated with the reference position $h_0(x) = \mathrm{id}_{I_0}(x) \equiv x$, so that the equilibrium mass located within the model interval $I = (a, b) \subset I_0 = [0, L]$ is

$$\int_I \rho_0 = \rho_0(b - a),$$

Find the spatial mass density $\rho = \rho(x, t)$ associated with the deformation $w = w(x, t)$. Hint:

$$\rho(x, t) = \lim_{(x_1, x_2) \to \{x\}} \frac{\rho_0(x_2 - x_1)}{w(x_2, t) - w(x_1, t)}.$$

(b) A simple way to model the internal forces associated with 1-D homogeneous deformation is using the Hooke's constant spring model familiar from elementary courses on ODEs. Find an expression for the internal force of compression/extension associated with homogeneous deformation. Hint(s): Assume a Hooke's constant k for which a mass at the end of the homogeneously deformed medium would experience a force $f = -k\,d$ where d is the displacement of the end from equilibrium. The force throughout the homogeneous deformation should be homogeneous. Your answer should admit a sign distinquishing compression from extension.

(c) Using your mass density formula from part **(a)** in the special case of a homogeneious deformation, express both the homogeneous deformation $h = h(x)$ and the force f from part **(b)** in terms of the density ratio ρ_0/ρ. Hint: Eliminate the constant c.

Fourier's law asserts that the thermal flux is proportional to the derivative of the temperature:

$$\Phi(x,t) = -c\ u_x(x,t) \qquad \text{or} \qquad \vec{\Phi}(\mathbf{x},t) = -c\ Du(\mathbf{x},t)$$

where c is a material constant called the **thermal conductivity**. The analogue for the internal motion modeled by the 1-D wave equation is the force relation

$$f(x,t) = \epsilon(1 - w_x(x,t)) \tag{1}$$

where ϵ is a material constant called the **elasticity**. The force $f = f(x,t)$ here represents the force experienced by the portion of the medium **to the right of position** x with a positive force associated with "push" to the right and a negative force associated with "pull" to the left. Notice $w_x(x,t) > 1$, i.e., local extension, is associated with pull while $w_x(x,t) < 1$, i.e., local compression, is associated with push. The sign may be reversed for modeling the force experienced by the portion of the medium to the left of position x according to Newton's first law—the law of equal and opposite reaction force.

Problem 4 (Hooke's constant and elasticity) Assume a time independent homogeneous deformation $w = h(x) = cx$ as in Problem 3.

(a) Calculate the force in this case from (1) and determine from your calcuation a relation between the Hooke's constant k associated with the homogeneous deformation of a spring and the elasticity ϵ. Hint: Find a formula for ϵ in terms of k assuming the force given in (1) is consistent with Hooke's law $f = -kd$.

(b) Is your calculation in part **(a)** above consistent if applied to a different length $\tilde{L}$ of the same spring "material." For example, if you take the spring of length L and then cut a piece of length $\tilde{L} < L$ from it, how does your calculation change/apply?

A generalization of Newton's second law appropriate for the derivation of the wave equation is the following: Given a subregion $I = (x_1, x_2) \subset [0, L]$ deformed to an interval $(w(x_1, t), w(x_2, t)) \subset \mathbb{R}$ at time t, there exists some $\xi \in (x_1, x_2)$ for which

$$m(x_1, x_2)\,\frac{d^2}{dt^2}w(\xi, t) = f(x_2, t) - f(x_1, t)$$

where

$$m(x_1, x_2) = \rho_0(x_2 - x_1)$$

is the mass associated with the deformed region $w(I)$.

Problem 5 Use the generalization of Newton's law along with the force relation (1) to derive the PDE

$$\rho_0 w_{tt} = \epsilon w_{xx}.$$

Problem 6 (scaling) Assume $w \in C^2((0,L)\times(0,\infty))$ satisfies the initial/boundary value problem

$$\begin{cases} \rho_0 w_{tt} = \epsilon w_{xx}, & (x,t) \in (0,L)\times(0,\infty) \\ w(0,t) = 0 \text{ and } w(L,t) = L, & t > 0 \\ w(x,0) = w_0(x), & 0 < x < L \\ w_t(x,0) = v_0(x), & 0 < x < L. \end{cases} \tag{2}$$

(a) Let $\tilde{w}(\xi, t) = w(\alpha\xi, t)$. Determine a value of α so that

$$\tilde{w}_{tt} = \tilde{w}_{\xi\xi}$$

and find the initial/boundary value problem satisfied by $\tilde{w}$.

(b) Let $\tilde{w}(x, \tau) = w(x, \alpha t)$. Determine a value of α so that

$$\tilde{w}_{\tau\tau} = \tilde{w}_{xx}$$

and find the initial/boundary value problem satisfied by $\tilde{w}$.

Problem 7 Consider the initial/boundary value problem (2) with $L = \pi$ and $w_0(x) = x + \sin x$.

(a) Make a plot with several points like those in Figure 1 representing the initial deformation $w_0(x)$.

(b) Check that $w(x,t) = x + \sin x \ \cos t$ satisfies (2) with $w_t(x,0) = v_0(x) =\equiv 0$.

(c) Animate the solution $w(x,t)$ plotting points as in part **(a)** with t as the animation parameter.

Problem 8 (factoring the wave operator) The **wave operator** is also called the d'Alembertian, and it is denoted by $\Box : C^2(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ with

$$\Box u = u_{tt} - u_{xx}.$$

Like the Laplacian $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $\Delta u = u_{xx}$ and the heat operator $L : C^2(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ by $Lu = u_t - u_{xx}$, the d'Alembertian is a second order partial differential operator.

Let $T : C^1(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ be the first order transport operator given by

$$Tu = u_t - u_x.$$

(a) Find a first order linear operator $S : C^1(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ for which

$$\Box u = S \circ T.$$

(b) Compute $T \circ S$.

(c) An operator $A : V \to W$ defined on a vector space V and taking values in a vector space W is **linear** if

$$A(cu) = cAu \qquad \text{for every } c \in \mathbb{R} \text{ and } u \in V$$

and

$$A(u+v) = Au + Av \qquad \text{for every } u, v \in V.$$

- **(i)** Show the wave operator is linear on $C^2(\mathbb{R} \times (0,\infty))$.
- **(ii)** Show the operator T is linear on $C^1(\mathbb{R} \times (0,\infty))$.
- **(iii)** Show the Laplace operator Δ is linear on $C^2((a,b))$ where $a, b \in \mathbb{R}$ with $a < b$.
- **(iv)** Show the heat operator L is linear on $C^2((a,b) \times (0,\infty))$ where $a, b \in \mathbb{R}$ with $a < b$.

Problem 9 (The wave equation on all of $\mathbb{R}$) Consider the initial value problem (IVP) for the wave equation:

$$\begin{cases} u_{tt} = u_{xx}, & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = 0, & x \in \mathbb{R} \end{cases} \tag{3}$$

where u_0 is a given function with $u_0 \in C^2(\mathbb{R})$. Assuming $u \in C^2(\mathbb{R}\times(0,\infty)) \cap C^0(\mathbb{R}\times[0,\infty))$ is a solution of (3) and $w = Tu \in C^1(\mathbb{R}\times(0,\infty)) \cap C^0(\mathbb{R}\times[0,\infty))$ where S and T are the factor operators from Problem 8, find an appropriate initial value problem for a transport equation satisfied by w.

In particular, find an appropriate initial value w_0 for the problem

$$\begin{cases} Sw = 0, & \text{on } \mathbb{R}\times(0,\infty) \\ w(x,0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

Problem 10 (first order linear equation) Consider the problem (4) with the initial condition you found in Problem 9. Solve that problem by completing the following steps:

(a) Consider a parameterized path $\gamma : [0,\infty) \to \mathbb{R}\times[0,\infty)$ given by $\gamma(t) = (\xi(t), t)$ for some real valued spatial function $x = \xi(t)$. Use the chain rule to compute

$$\frac{d}{dt} w \circ \gamma(t). \tag{5}$$

(b) Compare your result from (5) to the PDE from (4). Given any initial starting point $x_0 \in \mathbb{R}$ find an appropriate ODE for $\xi : [0,\infty) \to \mathbb{R}$ based on your comparison, and solve the ODE with the initial condition $\xi(0) = x_0$.

(c) With your solution for ξ from part **(b)** which should depend on x_0, consider for an arbitrary point $(x,t) \in \mathbb{R}\times(0,\infty)$ the equation

$$\gamma(t) = (x,t). \tag{6}$$

Choose x_0 so that (6) is satisfied.

(d) If you made the correct choice of ODE in part **(b)** you should now know the value of the quantity in (5), which should tell you

$$w(\xi(t),t) = w_0(x_0).$$

If you made the correct choice of w_0 in Problem 9 you should now know the solution of the problem (4) in terms of u_0 from the IVP in Problem 9.