Assignment 7: Elliptic and parabolic equations solutions
Corrected Version
Due Wednesday April 1, 2026

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Note on corrected version: There was an error in the formulation of the asymptotic condition(s) leading up to Problem 6 and important for Problems 7-9. Hopefully this corrected version has everything in order.

Problems1-9 below are about modeling the motion of mass using a mass density $\rho = \rho(\mathbf{x}, t)$ and a velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. Related material may be found in Section 2.5.3 of Haberman's book and in Haberman's Exercises 2.5.17-27. The approach is modeled on what we have covered on the heat equation and Laplace's equation.

Recall the mathematical model for conservation of thermal energy on a domain $\Omega \subset \mathbb{R}^n$:

$$\frac{d}{dt}\int_R \Theta = -\int_{\partial R} \Phi \cdot N \quad \text{for every } R \subset \Omega \tag{1}$$

where $\Theta = \Theta(\mathbf{x}, t)$ is a (scalar) thermal energy density and $\Phi = \Phi(\mathbf{x}, t)$ is a (vector valued) thermal energy flux field. Simplified versions of the key assumptions in deriving the heat equation $u_r = \Delta u$ may be useful for comparison:

1. (specific heat) The thermal energy density is (proportional to) the temperature u:
$$\Theta = u.$$

2. (Fourier's law) The thermal flux field is (proportional to) the spatial gradient of the temperature:
$$\Phi = -Du.$$

Finally, recall (and recall how to use) the main technical tools in the derivation:

Theorem 1 (divergence theorem) *If $V : \overline{R} \to \mathbb{R}^n$ is any vector field, then*

$$\int_{\partial R} V \cdot N = \int_R \operatorname{div} V.$$

Lemma 1 (fundamental lemma of vanishing integrals) *If $f \in C^0(\Omega)$ and*

$$\int_R f = 0 \qquad \textit{for every } R \subset \Omega,$$

then $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Omega$.

Similar terminology concerning time derivatives, spatial gradients, and Laplacian, etc. may be used for modeling the flow of mass. In this case, there is a simple and natural replacement for Fourier's law:

Problem 1 (mass flux and continuity) Given an open set Ω in $\mathbb{R}^n$, a mass density function $\rho : \Omega \times (0, \infty) \to \mathbb{R}$, and a velocity $\mathbf{v} : \Omega \times (0, \infty) \to \mathbb{R}^n$, assume the mass flux across a hypersurface $\mathcal{S}$ oriented by a unit normal field $N : \mathcal{S} \to \mathbb{R}^n$ is given by

$$\int_{\mathcal{S}} \rho \mathbf{v} \cdot N.$$

(a) Write down the analogue of Fourier's law implicit in the flux condition assumed here. Hint: The mass flux field is...

(b) Write down a version of (1) expressing the conservation of mass within a region $R \subset \Omega$ as a function of time in terms of ρ and V.

(c) Use the divergence theorem and the fundamental lemma to obtain the differential form of your conservation law from part **(b)**. Your answer should be a PDE involving ρ and $\mathbf{v}$. Note: The equation you obtain here is sometimes called the **continuity equation** or the **transport equation**, though the designation *transport equation* usually has a more specialized meaning in the study of partial differential equations: The "mathematical" transport equation in PDEs is $u_t + \mathbf{a} \cdot Du = 0$ where $\mathbf{a} \in \mathbb{R}^n$ is a constant vector and Du is the usual spatial gradient.

(d) Apply a product rule for the divergence to expand

$$\operatorname{div}(\rho \mathbf{v})$$

and write your PDE from part **(c)** in a form giving $\operatorname{div} \mathbf{v}$ in terms of $\mathbf{v}$ and derivatives of ρ. Your answer should require the assumption $\rho > 0$.

(e) Under what condition(s) does the continuity equation from part **(c)** have the form of a mathematical transport equation in PDE.

partial solution:

(a) Fourier's law: $\vec{\Phi} = -k\ Du$.

Mass motion/convection: $\vec{\Phi} = \rho\ \mathbf{v}$.

(b)

$$\frac{d}{dt}\int_R \rho = -\int_{\partial R} \rho\ \mathbf{v} \cdot N.$$

(c)

$$\int_R \rho_t = -\int_R \operatorname{div}(\rho\ \mathbf{v}).$$

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0.$$

(d)

$$\rho_t + D\rho \cdot \mathbf{v} + \rho \operatorname{div} \mathbf{v} = 0.$$

$$\operatorname{div} \mathbf{v} = -\frac{1}{\rho}(\rho_t + D\rho \cdot \mathbf{v}).$$

(e) When $\mathbf{v}$ is a constant vector. Then $\mathbf{v} = \mathbf{a}$ is the constant vector and ρ is the unknown function:

$$\rho_t + \mathbf{v} \cdot D\rho = 0$$

is a transport equation for ρ when $\mathbf{v}$ is constant.

Spatial versus Material Flow

The model for the flow of mass determined by a mass density ρ and a velocity field $\mathbf{v}$ may be considered in two different frameworks or from two different points of view:

(spatial framework) This is perhaps the usual or natural mathematical point of view. Here one "fixes" a point $\mathbf{x}$ in the "space" Ω and considers the usual spatial derivatives and time derivatives of, for example, ρ and $\mathbf{v}$ as well as other "quantities" $q = q(\mathbf{x}, t)$ which may be scalar or vector valued (or perhaps even tensor valued).

Examples of other scalar quantities might be the (kinetic) energy density

$$q(\mathbf{x}, t) = \frac{1}{2}\, \rho\, |\mathbf{v}|^2$$

or the pressure $p = p(\mathbf{x}, t)$.

An example of a vector quantity when $\Omega \subset \mathbb{R}^3$ might be the rotation or **vorticity** which is given in rectangular coordinates by

$$\vec{\omega}(\mathbf{x}, t) = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\, ,\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\, ,\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right). \tag{2}$$

(material framework) It is also natural in this model to fix a specific $\mathbf{y} \in \Omega$ and solve the IVP

$$\begin{cases} \mathbf{x}'(\tau) = \mathbf{v}(\mathbf{x}(\tau), \tau), \quad \tau \in \mathbb{R} \\ \mathbf{x}(0) = \mathbf{y} \end{cases}$$

for $\mathbf{x} = \mathbf{x}(t)$. The resulting path $\{\mathbf{x}(t) : t \in \mathbb{R}\} \subset \Omega$ is called the **flow line** through $\mathbf{y}$. Points on the flow line are denoted by $\phi(\mathbf{y}, t)$ so

$$\phi(\mathbf{y}, t) = \mathbf{x}(t).$$

The function $\phi : \Omega \times \mathbb{R} \to \Omega$ is called the **flow field function** or just the flow function.

With a flow function one may consider the value of a (spatial) quantity q **along the** (material) **flow** by restricting attention to the composition

$$q(\phi(\mathbf{y}, t), t) = q(\mathbf{x}(t), t).$$

The associated derivatives/rates of change

$$\frac{\partial}{\partial y_j} q(\phi(\mathbf{y},t),t) \qquad \text{and} \qquad \frac{\partial}{\partial t} q(\phi(\mathbf{y},t),t)$$

are called **material derivatives**.

Notice that while the flow of mass is modeled using both mass density ρ and a velocity field $\mathbf{v}$, the flow function is essentially based only on the velocity $\mathbf{v}$. For this reason the "flow" and many of its properties are sometimes identified only with the vector field $\mathbf{v}$. For example, a "flow" on a three dimensional domain is said to be irrotational if the vorticity vector defined in (2) vanishes; one also says $\mathbf{v}$ is irrotational in this case.

One tacitly assumes when a flow function is considered that the flow is persistent over all times $t \in \mathbb{R}$, so the particular initial time $t = 0$ assumes no particular distinction. Note: This also generally assumes existence and uniqueness for all $t \in \mathbb{R}$ for the initial value problem(s) defining the flow. We are working toward some special examples for which these assumptions are valid.

Problem 2 (backwards time and the semigroup property) Let $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be a flow function associated with the mass flow modeled in connection with $\rho : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^n$.

(a) Let $R = (0, L) \times (0, M)$ be a rectangular domain in $\mathbb{R}^2$ with $L, M > 0$. Solve the initial/boundary value problem for the heat equation

$$\begin{cases} u_t = \Delta u, & (x,y,t) \in R \times \mathbb{R} \\ u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,M,t) = 0, & (x,y,t) \in R \times \mathbb{R} \\ u(x,y,0) = \sin(\pi x/L)\sin(\pi y/M), & (x,y) \in R. \end{cases}$$

What happens to the temperature distribution as t tends to $-\infty$?

(b) How do the derivation of the continuity equation and the other aspects of Problem 1 change if ρ and $\mathbf{v}$ are functions defined for all $t \in \mathbb{R}$?

(c) Show

$$\phi(\phi(\mathbf{y},t_1),t_2) = \phi(\mathbf{y},t_1+t_2).$$

Hint: Use the existence and uniqueness theorem for the system(s) of ODEs defining the flow. Note: This identity is called the semigroup property for the flow.

Solution:

(a) Let $R = (0, L) \times (0, M)$ be a rectangular domain in $\mathbb{R}^2$ with $L, M > 0$. Solve the initial/boundary value problem for the heat equation

$$\begin{cases} u_t = \Delta u, & (x, y, t) \in R \times \mathbb{R} \\ u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, M, t) = 0, & (x, y, t) \in R \times \mathbb{R} \\ u(x, y, 0) = \sin(\pi x/L) \sin(\pi y/L), & (x, y) \in R. \end{cases}$$

We're going to get a solution of the form $u(x, y, t) = e^{-\alpha t} \sin(\pi x/L) \sin(\pi y/M)$ for some $\alpha > 0$. In fact,

$$\Delta u = -e^{-\alpha t} \left(\frac{\pi^2}{L^2} + \frac{\pi^2}{M^2} \right) \sin(\pi x/L) \sin(\pi y/M).$$

This means the solution is

$$u(x, y, t) = e^{-\left(\frac{\pi^2}{L^2} + \frac{\pi^2}{M^2} \right) t} \sin(\pi x/L) \sin(\pi y/M).$$

What happens to the temperature distribution as t tends to $-\infty$?

Answer: One has

$$\lim_{t \searrow -\infty} u(x, y, t) = \begin{cases} +\infty, & (x, y) \in R \\ 0, & (x, y) \in \partial R. \end{cases}$$

Of course the convergence (to $+\infty$) is not uniform on the rectangular domain R but takes the form of a large(r and larger) amplitude sine product distribution.

(b) How do the derivation of the continuity equation and the other aspects of Problem 1 change if ρ and $\mathbf{v}$ are functions defined for all $t \in \mathbb{R}$?

Answer: There are no changes; the designation of any particular time (positive or negative) is essentially relative...or irrelevant.

(c) Fix t_1 and consider the two functions $\mathbf{x}, \tilde{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{x}(\tau) = \phi(\phi(\mathbf{y}, t_1), \tau)$ and $\tilde{\mathbf{x}}(\tau) = \phi(\mathbf{y}, t_1 + \tau)$. Notice that

$$\mathbf{x}(0) = \phi(\phi(\mathbf{y}, t_1), 0) = \phi(\mathbf{y}, t_1) \qquad \text{and} \qquad \tilde{\mathbf{x}}(0) = \phi(\mathbf{y}, t_1).$$

Thus, both functions start with the same initial value. Next observe

$$\frac{d}{d\tau}\mathbf{x}(\tau) = \mathbf{v}(\phi(\phi(\mathbf{y}, t_1), \tau), \tau) = \mathbf{v}(\mathbf{x}(\tau), \tau).$$

This is the ODE satisfied by the flow function ϕ, in this case, on the outside. Note the ODE is independent of the initial condition.

On the other hand,

$$\frac{d}{d\tau}\tilde{\mathbf{x}}(\tau) = \frac{\partial \phi}{\partial t}(\mathbf{y}, t_1 + \tau) = \mathbf{v}(\phi(\mathbf{y}, t_1 + \tau), \tau) = \mathbf{v}(\tilde{\mathbf{x}}(\tau), \tau).$$

Here the initial condition shifts in the argument $\phi(\mathbf{y}, t_1 + \tau)$ but the ODE is again independent of the initial condition moving along in time in the field $\mathbf{v}$ only by time τ.

Thus, $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are both solutions of the IVP

$$\begin{cases} \mathbf{x}'(\tau) = \mathbf{v}(\mathbf{x}(\tau), \tau), \quad \tau \in \mathbb{R} \\ \mathbf{x}(0) = \phi(\mathbf{y}, t_1). \end{cases}$$

Thus, by existence and uniqueness for this IVP (assuming enough regularity so that an existence and uniqueness theorem holds) we get $\mathbf{x}(t_2) = \tilde{\mathbf{x}}(t_2)$ or

$$\phi(\phi(\mathbf{y}, t_1), t_2) = \phi(\mathbf{y}, t_1 + t_2).$$

Incompressibility

The flow of mass modeled in connection with $\rho : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^n$ is said to be **incompressible** if the material time derivative of the mass density vanishes:

$$\frac{\partial}{\partial t}\rho(\phi(\mathbf{y}, t), t) = 0 \qquad \text{for } (\mathbf{y}, t) \in \Omega \times \mathbb{R}. \tag{3}$$

Problem 3 (incompressibility) Show a flow is incompressible if and only if

$$\text{div}(\mathbf{v}) = 0.$$

Solution: We have taken the definition that incompressibility means

$$\frac{\partial}{\partial t}\rho(\phi(\mathbf{y}, t), t) = 0$$

in terms of the flow function. Expanding this derivative using the chain rule we get

$$\frac{\partial}{\partial t}\rho(\phi(\mathbf{y}, t), t) = D\rho(\phi(\mathbf{y}, t), t) \cdot \frac{\partial \phi}{\partial t}(\mathbf{y}, t) + \rho_t(\phi(\mathbf{y}, t), t) = 0.$$

Writing $\mathbf{x}(t) = \phi(\mathbf{y}, t)$ and using the flow IVP this can be written as

$$D\rho \cdot \mathbf{v} + \rho_t = 0.$$

On the other hand, we calculated in Problem 1 part **(d)** that

$$\text{div}\,\mathbf{v} = -\frac{1}{\rho}(\rho_t + D\rho \cdot \mathbf{v}).$$

Thus, incompressibility implies $\text{div}\,\mathbf{v} = 0$. On the other hand, if $\text{div}\,\mathbf{v} = 0$, then

$$\rho_t + D\rho \cdot \mathbf{v} = 0$$

because $\rho > 0$, and (3) follows from this.

Planar Flow and Rotation

For each of the remaining Problems 4-9 we restrict the spatial dimension to $n = 2$. Thus, strictly speaking we are only considering flows on planar domains $\Omega \subset \mathbb{R}^2$ below. It will be sometimes useful, however, to **extend** the flow associated with $\rho : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^2$ to a threee-dimensional flow associated with extensions

$$\overline{\rho} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad \text{by} \qquad \overline{\rho}(x, y, z, t) = \rho(x, y, t)$$

and

$$\overline{\mathbf{v}} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3 \qquad \text{by} \qquad \overline{\mathbf{v}}(x, y, z, t) = (v_1(x, y, t), v_2(x, y, t), 0).$$

Note that $\Omega \times \mathbb{R}$ is a cylindrical domain in $\mathbb{R}^3$.

Define the **rotation** $q = q(x, y, t)$ by

$$q(x, y, t) = \lim_{R \to \{(x,y)\}} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot T$$

where the limit is taken over "nice" subdomains $R \subset \Omega$ shrinking to $\{(x, y)\}$ and T denotes the counterclockwise unit tangent field on ∂R. For example, if $R = B_a(\mathbf{0})$, then $T(a \cos\theta, a \sin\theta) = (-\sin\theta, \cos\theta)$ for $\theta \in \mathbb{R}$. Counterclockwise rotation by $\pi/2$ is also generally denoted with a "perp" superscript so

$$(\cos\theta, \sin\theta)^{\perp} = (-\sin\theta, \cos\theta).$$

Problem 4 (rotation)

(a) Find a formula for the rotation in rectangular coordinates. Hint(s): Take $R = (x-\delta, x+\delta) \times (y-\epsilon, y+\epsilon)$ to be a rectangular domain and write

$$\int_R \mathbf{v}\cdot T = \sum_{j=1}^{4} \int_{E_j} \mathbf{v}\cdot T$$

where $E_1 = \{(\xi, y-\epsilon) : x-\delta < \xi < x+\delta\}$ is the bottom edge of R, E_2 is the right edge, and so on. You can express a difference like

$$v_1(\xi, y+\delta, t) - v_1(\xi, y-\delta, t)$$

using the mean value theorem:

$$\frac{v_1(\xi, y+\delta, t) - v_1(\xi, y-\delta, t)}{y+\delta-(y-\delta)} = \frac{\partial v_1}{\partial y}(\xi, \eta_*, t)$$

for some $\eta_* = \eta_*(\xi, y, \epsilon)$ with $y-\delta < \eta_* < y+\delta$.

(b) Show the scalar rotation $q : \Omega \times \mathbb{R} \to \mathbb{R}$ of $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^2$ vanishes if and only if the vorticity of the extension $\overline{\mathbf{v}} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ defined in (2) is a vanishing vector field.

Stream Functions and Irrotational Flow

The velocity field $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^2$ is said to be **irrotational** if the rotation defined above vanishes.

A function $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is called a **stream function** for $\mathbf{v}$ if

$$\mathbf{v}^\perp = Du. \tag{4}$$

The condition (4) may be compared to Fourier's law $\Phi = -Du$ which says the thermal flux field is a gradient field. Unlike Fourier's law the condition (4) is not interpreted as a model for some physical mechanism or "law." Instead a particular flow may admit a stream function or may not admit a stream function.

A stream function, when it exists, is traditionally denoted by ψ, so that

$$\mathbf{v}^\perp = (-v_2, v_1) = D\psi.$$

Note: We have previously used ψ to denote the polar coordinates map. The stream function has nothing inherent to do with the polar coordinates map, though we are going to use polar coordinates below. (We will avoid calling the polar coordinates map ψ in this context to avoid confusion.)

Problem 5 (stream function and irrotational flow)

(a) Show that if $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^2$ admits a stream function, then the flow is incompressible.

(b) Find the typo on Haberman's Exercise 2.5.20 (Fourth Edition).

(c) Show that if an irrotational flow admits a stream function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$, then ψ is spatially harmonic:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

Asymptotic Conditions and Example Flows

For each of the remaining Problems 6-9 we restrict to a specfic domain

$$\Omega = \mathbb{R}^2 \backslash \overline{B_a(\mathbf{0})}$$

where $a > 0$. This is an exterior domain which we consider relative to the domain

$$R = (a, \infty) \times \mathbb{R}$$

with respect to polar coordinates. We assume $\mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^2$ satisfies the following:

1. $\mathbf{v}$ admits a stream function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ with

$$\mathbf{v}^\perp = D\psi,$$

2. $\mathbf{v}$ is irrotational so that ψ is spatially harmonic and

$$\Delta \psi = 0, \qquad \text{and}$$

3. $\mathbf{v}$ is asymptotic to a constant speed horizontal velocity field $\mathbf{v}_\infty = -s(1,0)$ where $s > 0$ is the constant speed of the far-field flow. This is also the vector field corresponding the flow function $\phi(\mathbf{x},t) = \mathbf{x} - ts(1,0)$. This assumption is intended to model the motion of the disk horizontally to the right at a constant speed s in an ambient fluid with zero velocity in the limit $x^2 + y^2 \to \infty$.

The third condition/assumption requires some further explanation. Roughly speaking we mean

$$\lim_{x^2+y^2\to\infty} |\mathbf{v}(x,y,t) - \mathbf{v}_\infty| = 0.$$

But more precisely we mean that if $w = w(r,\theta,t) = \psi(r\cos\theta, r\sin\theta, t)$ has a Fourier expansion

$$w = \sum_{j=0}^{\infty} w_j(r,t)\ \cos(j\theta) + \sum_{j=1}^{\infty} W_j(r,t)\ \sin(j\theta) \tag{5}$$

and $w_\infty = w_\infty(r,\theta)$ has a Fourier expansion

$$w_\infty = \sum_{j=0}^{\infty} c_j(r)\ \cos(j\theta) + \sum_{j=1}^{\infty} C_j(r)\ \sin(j\theta), \tag{6}$$

then the asymptotic condition on the field $\mathbf{v}$ requires the convergence of the Fourier coefficients of Dw and Dw_∞. (Recall that $\mathbf{v} = -D\psi^\perp$. Requiring $\mathbf{v}$ is asymptotic to $\mathbf{v}_\infty$ means requiring $D\psi$ is asymptotic to $D\psi_\infty$ which is equivalent to requiring Dw is asymptotic to Dw_∞.) More precisely, note that differentiating termwise gives

$$w_r = \sum_{j=0}^{\infty} \frac{\partial w_j}{\partial r}(r,t)\ \cos(j\theta) + \sum_{j=1}^{\infty} \frac{\partial W_j}{\partial r}(r,t)\ \sin(j\theta), \tag{7}$$

$$w_\theta = -\sum_{j=1}^{\infty} j w_j(r,t)\ \sin(j\theta) + \sum_{j=1}^{\infty} j W_j(r,t)\ \cos(j\theta), \tag{8}$$

$$(w_\infty)_r = \sum_{j=0}^{\infty} c_j'(r)\ \cos(j\theta) + \sum_{j=1}^{\infty} C_j'(r)\ \sin(j\theta), \tag{9}$$

and

$$(w_\infty)_\theta = -\sum_{j=0}^{\infty} j c_j(r)\ \sin(j\theta) + \sum_{j=1}^{\infty} j C_j(r)\ \cos(j\theta). \tag{10}$$

Equating coefficients asymptotically we get the desired meaning of $\mathbf{v} \sim \mathbf{v}_\infty$ as $r \to \infty$:

$$\lim_{r\to\infty} \left| \frac{w_j}{\partial r}(r,t) - c_j'(r) \right| = 0, \qquad j = 0, 1, 2, 3, \ldots, \tag{11}$$

$$\lim_{r\to\infty} \left| \frac{W_j}{\partial r}(r,t) - C_j'(r) \right| = 0, \qquad j = 1, 2, 3, \ldots, \tag{12}$$

$$\lim_{r\to\infty} |w_j(r,t) - c_j(r)| = 0, \qquad j = 1, 2, 3, \ldots \tag{13}$$

(notice the $j = 0$ term has dropped out here because the derivative of the constant $\cos(0\theta) = 1$ vanishes), and

$$\lim_{r\to\infty} |W_j(r,t) - C_j(r)| = 0, \qquad j = 1, 2, 3, \ldots. \tag{14}$$

Note: The absolute values can be left off in these limits.
Note on corrected version: The conditions (11-14) above constitute the main correction. Previously I had equated (asymptotically) the coefficients in the Fourier series of w and w_∞ without taking the derivatives. By making this error I obtained the conditions (13) and (14) but with condition (13) also holding for $j = 0$. The problem was not so much that I left off the conditions (11) and (12) but that the additional condition (13) also holding for $j = 0$ ruled out the term which eventually models the rotation of the circle/cylinder in the examples below.

The point is that it is the vector fields $\mathbf{v}$ and $\mathbf{v}_\infty$ which are supposed to be asymptotic, so that means not the stream functions ψ and ψ_∞ themselves or the poloar coordinate versions w and w_∞ but the derivatives/gradients Dw and Dw_∞.

Problem 6 (PDE and finite boundary condition) The function w described above is also called a stream function.

(a) Find the PDE satisfied by the stream function w.

(b) Use the condition that no mass crosses $\partial B_a(\mathbf{0})$ to obtain the boundary condition

$$w_\theta(a,\theta) = \frac{\partial w}{\partial \theta}(a,\theta) = 0, \qquad \theta \in \mathbb{R}.$$

(c) Show by a normalization one may assume the homogeneous Dirichlet condition

$$w(a,\theta) = 0, \qquad \theta \in \mathbb{R}.$$

Solution:

(a) We have $w(r, \theta, t) = \psi(r \cos\theta, r \sin\theta, t)$ and $\Delta\psi = 0$.

$$
\begin{aligned}
w_r &= \psi_x \; \cos\theta + \psi_y \; \sin\theta \\
w_\theta &= r \; \psi_x \; (-\sin\theta) + r \; \psi_y \; \cos\theta \\
&= r(-\psi_x \; \sin\theta + \psi_y \; \cos\theta) \\
w_{rr} &= \psi_{xx} \; \cos^2\theta + 2\psi_{xy} \; \cos\theta \sin\theta + \psi_{yy} \; \sin^2\theta \\
w_{r\theta} &= r(-\psi_{xx} \; \cos\theta \sin\theta - \psi_{xy} \; \sin^2\theta + \psi_{xy} \; \cos^2\theta + \psi_{yy} \; \cos\theta \sin\theta) \\
&\qquad - \psi_x \; \sin\theta + \psi_y \; \cos\theta \\
w_{\theta\theta} &= r^2(\psi_{xx} \; \sin^2\theta + -2\psi_{xy} \; \cos\theta \sin\theta + \psi_{yy} \; \cos^2\theta) \\
&\qquad - r(\psi_x \; \cos\theta + \psi_y \; \sin\theta).
\end{aligned}
$$

Therefore,

$$
w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}.
$$

(b) The effective flux field is $\rho\mathbf{v}$ so we must have $\mathbf{v} \cdot N = 0$. That is $D\psi^\perp \cdot N = 0$. Notice the system

$$
\begin{aligned}
w_r &= \psi_x \; \cos\theta + \psi_y \; \sin\theta \\
w_\theta &= -r(\psi_x \; \sin\theta + \psi_y \; \cos\theta)
\end{aligned}
$$

implies

$$
\begin{aligned}
\psi_x &= \frac{1}{r}(w_r \; \cos\theta - w_\theta \; \sin\theta) \\
\psi_y &= \frac{1}{r}(w_r \; \sin\theta + w_\theta \; \cos\theta).
\end{aligned}
$$

Thus,

$$
(-\psi_y, \psi_x) = \frac{1}{r}(-(w_r \; \sin\theta + w_\theta \; \cos\theta), w_r \; \cos\theta - w_\theta \; \sin\theta).
$$

Along $\partial B_a(\mathbf{0})$ where $N = (\cos\theta, \sin\theta)$ and $w_\theta = w_\theta(a, \theta)$ we have

$$
0 = \frac{1}{a}(-(w_r \; \sin\theta + w_\theta \; \cos\theta), w_r \; \cos\theta - w_\theta \; \sin\theta) \cdot (\cos\theta, \sin\theta) = -\frac{w_\theta}{a}.
$$

(c) If $w_\theta = 0$ on $\partial B_a(\mathbf{0})$, we can say

$$\frac{\partial}{\partial\theta}\psi(a\cos\theta, a\sin\theta) = 0,$$

and θ is a parameter for the boundary circle, this means ψ is constant on that boundary circle. By subtracting a constant which doesn't change the Laplacian, we can assume the constant is zero, that is,

$$w\Big|_{r=a} = \psi\Big|_{\partial B_a(\mathbf{0})} \equiv 0.$$

Problem 7 (asymptotic condition) The function w_∞ described above is also called a stream function for the constant velocity field.

(a) Find an explicit formula for a stream function w_∞. (The answer here is not unique.)

(b) Determine the Fourier expansion of the form (6) for the function you found in part **(a)**.

(c) Derive specific limiting/asymptotic conditions on the Fourier coefficients w_j and W_j in the Fourier expansion for the stream function w of the form given in (5) above based on (11-14).

Solution:

(a) We need here $D\psi = \mathbf{v}_\infty^\perp = (-s, 0)^\perp$. That is,

$$D\psi = (\psi_x, \psi_y) = (0, -s).$$

Therefore, we can take $\psi = -sy$. This gives

$$w_\infty(r,\theta) = -sr\sin\theta.$$

(b) This is already a Fourier expansion of the desired form

$$w_\infty = \sum_{j=0}^{\infty} c_j(r)\ \cos(j\theta) + \sum_{j=1}^{\infty} C_j(r)\ \sin(j\theta)$$

with one term and $C_1(r) = -sr$. All other coefficients are zero.

(c) The Fourier expansion

$$w = \sum_{j=0}^{\infty} w_j(r,t)\ \cos(j\theta) + \sum_{j=1}^{\infty} W_j(r,t)\ \sin(j\theta)$$

given in (5) has gradient asymptotic to that of $w_\infty(r,\theta) = -sr\sin\theta$ if

$$\lim_{r\to\infty} \frac{\partial w_j}{\partial r}(r,t) = 0 \qquad \text{for} \qquad j = 0,1,2,3,\ldots,$$

$$\lim_{r\to\infty} \left(\frac{\partial W_1}{\partial r}(r,t) + s \right) = 0,$$

$$\lim_{r\to\infty} \frac{\partial W_j}{\partial r}(r,t) = 0 \qquad \text{for} \qquad j = 2,3,4,\ldots,$$

$$\lim_{r\to\infty} w_j(r,t) = 0 \qquad \text{for} \qquad j = 1,2,3,\ldots,$$

$$\lim_{r\to\infty} (W_1(r,t) + sr) = 0,$$

and

$$\lim_{r\to\infty} W_j(r,t) = 0 \qquad \text{for} \qquad j = 2,3,4,\ldots.$$

Example Flows

For each of the remaining Problems 8-9 we restrict to **steady flows** with (spatial) velocity field $\mathbf{v}$ (and stream functions ψ and $\mathbf{w}$) independent of time t.

Problem 8 (example flows)

(a) Appending to the conditions you have derived in Problems 6-7 above the periodicity conditions

$$w(r,\theta+2\pi) = w(r,\theta), \qquad (r,\theta) \in R$$
$$w_\theta(r,\theta+2\pi) = w_\theta(r,\theta), \qquad (r,\theta) \in R,$$

use a separation of variables $w(r,\theta) = A(r)B(\theta)$ to find a superposition of the form

$$w(r,\theta) = \sum_{j=0}^{\infty} w_j(r)\ \cos(j\theta) + \sum_{j=1}^{\infty} W_j(r)\ \sin(j\theta)$$

satisfying all the conditions you derived in Problems 6-7. Up to an additive constant your answer should contain one other arbitrary constant with

$$w_0(r) = c\ln(r/a).$$

In this way, you should obtain a one parameter family of example flows for each fixed speed s.

(b) Define the boundary rotation to be

$$Q_\partial = \int_{\partial B_a(\mathbf{0})} \mathbf{v}\cdot T$$

where T is the counterclockwise unit tangent vector field along $\partial B_a(\mathbf{0})$, and show the value is given by $-2\pi c$ where c is the constant determining the "constant" term w_0 in the Fourier expansion of w.

(c) Explain how the constant c can be used to model the rotation of the disk/cylinder. For example explain how clockwise/counterclockwise rotation should be related to the velocity on $\partial B_a(\mathbf{0})$ and the boundary rotation.

Solution:

(a) With $w(r,\theta) = A(r)B(\theta)$ and $r(rw_r)_r + w_{\theta\theta} = 0$, we get

$$r(rA')'B = -AB'' \qquad \text{and} \qquad \frac{r(rA')'}{A} = -\frac{B''}{B} = \lambda.$$

The periodic boundary conditions with the irregular Sturm-Liouville problem for B is familiar giving $B_j(\theta) = \cos(j\theta)$ and $B_j(\theta) = \sin(j\theta)$ with $\lambda_j = j^2$ for $j = 0, 1, 2, 3, \ldots$.

With this, we also get the equidimensional ODE $r^2 A_j'' + rA_j' - j^2 A_j$. At $r = a$ we get $A(a) = 0$ in accord with Problem 6.

Out at $r = \infty$ we get something more complicated addressed by the asymptotics of Problem 7. Let's leave the asymptotics alone for a moment and solve the ODE:

In the case $j = 0$ corresponding to the constant cosine mode $B_0(\theta) = \cos(0) \equiv 1$, we have

$$A_0'' = -\frac{A_0'}{r}.$$

Trying $A_0(r) = r^\alpha$ does give one solution $A_0 = c$ (constant), but I'll get a full basis by solving the ODE as a separable equation in A_0', so that

$$\frac{A_0''}{A_0'} = -\frac{1}{r} \qquad \text{so that} \qquad \ln A_0' - \ln A_0'(a) = -\ln r + \ln a.$$

Then exponentiating we get

$$\frac{A_0'}{A_0'(a)} = \frac{a}{r} \qquad \text{so that} \qquad A_0 = A_0(a) + A_0'(a) a \ln r.$$

Thus, with $B_0 \equiv 1$, this is the associated separated variables solution.

The boundary condition $A_0(a) = 0$ implies we can write $A_0 = w_0 = c\ln(r/a)$ with $c = aA_0'(a)$.

For the other Fourier modes we try $A = A_j = r^\alpha$ and get

$$(\alpha(\alpha - 1) + \alpha - j^2) r^\alpha = 0 \qquad \text{or} \qquad \alpha = \pm j.$$

In this case, the solution associated with the negative exponent is not ruled out because we don't get to $r = 0$. In fact, the other positive exponent $\alpha = j$

giving r^j will be the problem one because we need to worry about $r \to \infty$ where tame/constant/horizontal asymptotics come into play. In any case, we get some solutions

$$A_j(r) = b_j r^j + \frac{\tilde{b}_j}{r^j} \qquad \text{for} \qquad j = 1, 2, 3, \ldots$$

where b_j and $\tilde{b}_j$ are some constants. The condition $A_j(a) = 0$ implies

$$b_j a^j = -\frac{\tilde{b}_j}{a^j} \qquad \text{or} \qquad \tilde{b}_j = -b_j a^{2j}.$$

In this way we see

$$A_j(r) = b_j \left(r^j - \frac{a^{2j}}{r^j} \right).$$

In obtaining the time independent Fourier expansion

$$w = \sum_{j=0}^{\infty} w_j(r)\ \cos(j\theta) + \sum_{j=1}^{\infty} W_j(r)\ \sin(j\theta)$$

given in (5) we need more symbols for the constants, so we write

$$w_j(r) = b_j \left(r^j - \frac{a^{2j}}{r^j} \right) \qquad \text{and} \qquad W_j(r) = \beta_j \left(r^j - \frac{a^{2j}}{r^j} \right).$$

This means our final superposition looks like

$$w(r, \theta) = c \ln(r/a) + \sum_{j=1}^{\infty} b_j \left(r^j - \frac{a^{2j}}{r^j} \right) \cos(j\theta) + \sum_{j=1}^{\infty} \beta_j \left(r^j - \frac{a^{2j}}{r^j} \right) \sin(j\theta).$$

Finally, to this expression we apply the (time independent) asymptic conditions

$$\lim_{r \to \infty} w_j'(r) = 0 \qquad \text{for} \qquad j = 0, 1, 2, 3, \ldots,$$

$$\lim_{r \to \infty} (W_1'(r) + s) = 0,$$

$$\lim_{r \to \infty} W_j'(r) = 0 \qquad \text{for} \qquad j = 2, 3, 4, \ldots,$$

$$\lim_{r \to \infty} w_j(r) = 0 \qquad \text{for} \qquad j = 1, 2, 3, \ldots,$$

$$\lim_{r \to \infty} (W_1(r) + sr) = 0,$$

and

$$\lim_{r\to\infty} W_j(r) = 0 \qquad \text{for} \qquad j = 2, 3, 4, \ldots$$

of Problem 7. The last of these conditions implies

$$\lim_{r\to\infty} \beta_j \left(r^j - \frac{a^{2j}}{r^j} \right) = 0 \qquad \text{for} \qquad j = 2, 3, 4, \ldots$$

which means $\beta_j = 0$ for $j = 2, 3, 4, \ldots$. Note this forces the corresponding conditions on the derivatives to be automatically satisfied.

Working our way up through the asymptotic conditions we need

$$\lim_{r\to\infty} \left(\beta_1 \left(r - \frac{a^2}{r} \right) + sr \right) = 0.$$

This requires $\beta_1 = -s$.

The next conditions require

$$\lim_{r\to\infty} b_j \left(r^j - \frac{a^{2j}}{r^j} \right) = 0 \qquad \text{for} \qquad j = 1, 2, 3, \ldots.$$

This only happens if $b_j = 0$ for $j = 1, 2, 3, \ldots$. This means the corresponding conditions on the derivatives are automatically satisfied, but it is important to note the index $j = 0$ is omitted here.

Continuing up the list, the next conditions are derivative conditions on W_j' for $j = 2, 3, 4, \ldots$ which are automatically satisfied.

The next condition is

$$\lim_{r\to\infty} \left(W_1'(r) + s \right) = 0 \qquad \text{or} \qquad \lim_{r\to\infty} W_1'(r) = -s$$

which is also more or less automatically satisfied because

$$W_1(r) = \beta_1 \left(r - \frac{a^2}{r} \right)$$

and we have taken $\beta_1 = -s$. This means

$$W_1'(r) = -s \left(1 + \frac{a^2}{r^2} \right)$$

and this has the required limit.

Finally, the very first condition

$$\lim_{r\to\infty} w_j'(r) = 0 \qquad \text{for} \qquad j = 0$$

is the only one remaining to check. We have taken $w_0(r) = c\ \ln(r/a)$. This means

$$w_0'(r) = \frac{c}{r},$$

and indeed

$$\lim_{r\to\infty} \frac{c}{r} = 0.$$

Thus, as claimed the superposition solutions for the problem are given by

$$w(r,\theta) = c\ln(r/a) - s\left(r - \frac{a^2}{r}\right)\sin\theta$$

and depend on one arbitrary constant. Thus, for each fixed speed s in the far-field, we obtain one example solution/flow.

(b) Note that

$$w_r = \frac{c}{r} - s\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

and

$$w_\theta = -s\left(r - \frac{a^2}{r}\right)\cos\theta$$

It follows from the calculations in part **(b)** of the solution of Problem 6 above that

$$\psi_x = \frac{cx}{x^2+y^2} - s\frac{2a^2xy}{(x^2+y^2)^2}$$

and

$$\psi_y = \frac{cy}{x^2+y^2} - s\left(1 - \frac{a^2}{x^2+y^2} - \frac{2a^2y^2}{(x^2+y^2)^2}\right).$$

One can also start with

$$\psi = c\ln\left(\frac{1}{a}\sqrt{x^2+y^2}\right) - sy\left(1 - \frac{a^2}{x^2+y^2}\right)$$

and compute

$$\psi_x = \frac{cx}{x^2+y^2} - s\frac{2a^2xy}{(x^2+y^2)^2}$$

and

$$\psi_y = \frac{cy}{x^2+y^2} - s\left(1 - \frac{a^2}{x^2+y^2}\right) - s\frac{2a^2y^2}{(x^2+y^2)^2}.$$

In any case $\mathbf{v} = -D\psi^\perp = -(-\psi_y, \psi_x) = (\psi_y, -\psi_x)$, so

$$\mathbf{v} = \left(\frac{cy}{x^2+y^2} - s\left(1 - \frac{a^2}{x^2+y^2} + \frac{2a^2y^2}{(x^2+y^2)^2}\right), -\frac{cx}{x^2+y^2} + s\frac{2a^2xy}{(x^2+y^2)^2}\right)$$

and when $(x, y) \in \partial B_a(\mathbf{0})$ we have $T = (-y, x)/a$. Therefore, the rotation is given by

$$\begin{aligned} Q_\partial &= \int_{\partial B_a(\mathbf{0})} \mathbf{v}\cdot T \\ &= \frac{1}{a}\int_{\partial B_a(\mathbf{0})} \left(-\frac{cy^2}{x^2+y^2} + s\left(y - \frac{a^2y}{x^2+y^2} - \frac{2a^2ys}{(x^2+y^2)^2}\right)\right. \\ &\qquad\qquad \left. -\frac{cx^2}{x^2+y^2} + s\ \frac{2a^2x^2y}{(x^2+y^2)^2}\right). \end{aligned}$$

Notice that an integral with an integrand which is odd in either x or y will vanish when integrated on the circle $\partial B_a(\mathbf{0})$. For example,

$$\int_{\partial B_a(\mathbf{0})} x = \int_{\partial B_a(\mathbf{0})} y = \int_{\partial B_a(\mathbf{0})} x^3 = \int_{\partial B_a(\mathbf{0})} y^3 = 0.$$

This applies to several of the terms in the integration, so we get

$$\begin{aligned} Q_\partial &= \frac{1}{a}\int_{\partial B_a(\mathbf{0})} \left(-\frac{cy^2}{x^2+y^2} - \frac{cx^2}{x^2+y^2}\right) \\ &= -\frac{1}{a}\int_{\partial B_a(\mathbf{0})} c \\ &= -2\pi c. \end{aligned}$$

(c) Remember this is modeling a circle or cylinder moving to the right in a stationary fluid. With that in mind, consider the velocities at the compass points for different values of c:

$$\mathbf{v}(a,0) = \left(0, -\frac{c}{a}\right)$$

$$\mathbf{v}(0,a) = \left(\frac{c}{a} - 2s, 0\right)$$

$$\mathbf{v}(-a,0) = \left(0, \frac{c}{a}\right)$$

$$\mathbf{v}(0,-a) = \left(-\frac{c}{a} - 2s, 0\right).$$

Notice first the leading edge at $(a, 0)$. Here the velocity is vertical and has the opposite sign of c. In particular, when $c > 0$, the velocity at the leading edge is pointing down. This would presumably correspond to a clockwise rotation of the circle/cylinder increasing the velocity in the clockwise direction.

Let's see if this assumption is consistent. At the top we have $\mathbf{v}(0, a) = (-2s, 0)$ when $c = 0$, which makes sense, and if the cylinder is rotating clockwise this value should be reduced (or less to the left) when $c > 0$. It is due to the addition of the positive quantity c/a in the horizontal component:

$$\frac{c}{a} - 2s > -2s.$$

At the bottom, we expect increased velocity from the situation when $c = 0$ in which $\mathbf{v}(0, -a) = (-2s, 0)$ which is the same as at the top. This is indeed what we see with

$$\mathbf{v}(0,-a) = \left(-\frac{c}{a} - 2s, 0\right)$$

having greater magnitude (and still pointing to the left).

Finally, at the trailing edge we would expect an upward vertical velocity vector if the circle is rotating clockwise, and for $c > 0$ we get

$$\mathbf{v}(-a,0) = \left(0, \frac{c}{a}\right)$$

which is what we expect.

According to Bernoulli's law greater velocity corresponds to lower pressure. This is how an airfoil is supposed to work. When there is greater velocity over the top than under the bottom, then there is lower pressure at the top than under the bottom, and one gets/expects lift. Normally, this difference in velocities and corresponding difference in pressures is achieved by changing the shape of the airfoil. We're not going to get that here becuase we have a (too) simple cross-section which is a disk. On the other hand, we can induce a velocity difference by rotating the disk. It is not a usual design choice for an airplane wing to rotate. Furthermore even for a strange airplane with rotating cylinders for wings, we do not seem to get something suggesting lift (at least in terms of the velocity difference) for clockwise rotation with $c > 0$. When $c > 0$ at least the velocity $\mathbf{v}(0, a)$ at the top point is always smaller in magnitude than the velocity $\mathbf{v}(0, -a)$ at the bottom point:

$$2s - \frac{c}{a} < 2s + \frac{c}{a}.$$

Notice these are the magnitudes for $c > 0$ and c "small" in the sense that $c < 2as$. There is a critical rotation rate $c = 2as$ at which the velocity $\mathbf{v}(0, a)$ vanishes. In this case, the stagnation point starting at the leading edge $(a, 0)$ when $c = 0$ has moved to the top.

The magnitude inequality comparing the top velocity to the bottom for $c \geq 2as$ should be

$$\frac{c}{a} - 2s < 2s + \frac{c}{a}.$$

One never gets any lift (or can expect lift let's say) when $c > 0$. When $c < 0$ one expects a net downward force on the circle/cylinder. But for counterclockwise rotation $c < 0$, the velocity discussion is reversed and the model can predict lift.

Problem 9 (stream lines and lift) When a velocity field $\mathbf{v}$ admits a stream function ψ, then the flow lines are called stream lines.

(a) Show the level curves of ψ are the flow lines associated with $\mathbf{v}$.

(b) Plot the stream lines for some of your example flows with $a = s = 1$ and various values of c for example.

(c) The **boundary force** or the force per unit length on the cylinder $B_a(\mathbf{0}) \times \mathbb{R} \subset \mathbb{R}^3$ in the three-dimensional extension flow can be modeled by the vector valued force field

$$\mathbb{F} = -\int_{(x,y) \in \partial B_a(\mathbf{0})} \left(p_0 - \frac{1}{2}\rho |\mathbf{v}|^2 \right) (x, y).$$

The scalar quantity

$$p = p_0 - \frac{1}{2}\rho |\mathbf{v}|^2$$

is Bernoulli's model/approximation for the pressure where p_0 is constant and usually ρ is assumed constant as well. Notice this pressure is lower when the velocity has larger magnitude.

(i) Show the horizontal component of $\mathbb{F}$ is zero. This is called the **drag**.

(ii) Show the vertical component of $\mathbb{F}$ is nonzero precisely when $c \neq 0$ and determine which rotation direction of the disk/cylinder results in a positive value (i.e. **lift**).

Problem 10 (uniqueness for Poisson's equation) Let v_1 and v_2 be C^2 solutions of the Dirichlet boundary value problem

$$\begin{cases} \Delta v = f, & \text{on } \Omega \\ v\big|_{\partial\Omega} = g, & \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, the function $f \in C^0(\Omega)$ and $g : \partial\Omega \to \mathbb{R}$. Show $v_1 \equiv v_2$ so that this problem has a unique solution. Hint(s): Find a boundary value problem satisfied by the difference $u = v_1 - v_2$ and apply the (weak) maximum principle.

Solution: The boundary value problem satisfied by $u = v_1 - v_2$ in this case is

$$\begin{cases} \Delta u = 0, & \text{on } \Omega \\ u\big|_{\partial\Omega} = 0. & \end{cases}$$

Since u is harmonic we must have by the weak maximum principle that $u \leq 0$ where

$$0 = \max_{\partial\Omega} u.$$

Since $-u$ is also harmonic, we have $-u \leq 0$ as well, or $u \geq 0 \ldots$ or

$$0 \leq u \leq 0.$$

That is, $v_1 \equiv v_2$.