Assignment 7: Review of Elliptic and Parabolic Equations

Pace: Thursday October 31, 2024, Due Tuesday November 5, 2024

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Problem 1 (Green's theorem; Haberman Exercise 1.5.7)

(a) State Green's theorem. (Look it up and be sure you understand what it says if necessary.)

(b) Derive the heat equation in two dimensions using Green's theorem. Hint: Rotate your vector fields on ∂R by an angle $\pi/2$.

Problem 2 (1-D heat equation, Haberman 2.4.1)

(a) Solve the initial/boundary value problem for the 1-D heat equation

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\pi)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(\pi,t), & t > 0 \\ & \\ u(x,0) = \sin x, & 0 < x < \pi. \end{cases}$$

(b) Use mathematical software to plot the graph of your solution.

(c) Use mathematical software to produce a time animation of your solution.

Problem 3 (divergence) Let U be an open subset of $\mathbb{R}^2$ and assume $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field. Assume also that the coordinate functions v_1 and v_2 of $\mathbf{v} = (v_1, v_2)$ have continuous first partial derivatives on U. Take $\mathbf{p} = (p_1, p_2) \in U$ and consider for $\epsilon, \delta > 0$ a rectangular domain

$$R = (p_1 - \epsilon, p_1 + \epsilon) \times (p_2 - \delta, p_2 + \delta) = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| < \epsilon \text{ and } |x_2 - p_2| < \delta\}.$$

Finally, assume the closure

$$\overline{R} = [p_1 - \epsilon, p_1 + \epsilon] \times [p_2 - \delta, p_2 + \delta] = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| \le \epsilon \text{ and } |x_2 - p_2| \le \delta\}$$

satisfies $\overline{R} \subset U$.

(a) Express the boundary integral

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \sum_{j=1}^{4} I_j$$

where $\mathbf{n}$ is the outward unit normal field on ∂R as the sum of four elementary integrals of the form

$$I = \int_a^b f(t)\, dt$$

each corresponding to a single side of ∂R. Be careful to express the integrals I_j for $j = 1, 2, 3, 4$ precisely and in full detail so that the dependence of the arguments of v_1 and v_2 on the variable t and the lengths ϵ and δ is clearly indicated.

(b) Combine the integrals from part **(b)** above in pairs corresponding to opposite sides, and apply the mean value theorem to the resulting integrands. Hint: If the segment

$$\{(a, y) : y_1 \le y \le y_2\}$$

is a subset of U, then by the mean value theorem one can write

$$v_1(a, y_2) - v_1(a, y_1) = (y_2 - y_1)\frac{\partial v_1}{\partial y}(a, y_*)$$

for some y_* with $y_1 < y_* < y_2$.

(c) Use your expressions for part **(b)** to compute the following limits

(i)

$$\lim_{\epsilon \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(ii)

$$\lim_{\delta \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iii)

$$\lim_{\epsilon \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iv)

$$\lim_{\delta \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(d) The mean value theorem for integrals states that if f is continuous on the closed interval $[a, b]$, then there is some $x_{**} \in (a, b)$ for which

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(x_{**}).$$

Use this result along with your expression from part **(b)** above to write

$$\frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}$$

as a sum of two terms in which no integrals appear.

(e) Compute the limits

(i)

$$\lim_{\epsilon \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(ii)

$$\lim_{\delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iii)

$$\text{div}\, \mathbf{v}(\mathbf{p}) = \lim_{\epsilon, \delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

Problem 4 (uniqueness for solutions of the heat equation) Let u and v be solutions of the inital/boundary value problem

$$\begin{cases} u_t = \Delta u + f, & (\mathbf{x}, t) \in U \times (0, \infty) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in U \\ u(\mathbf{x}, t) = u_0(\mathbf{x}), & x \in \partial U, t > 0 \end{cases}$$

on the open, bounded, connected spatial domain $U \subset \mathbb{R}^n$ with smooth boundary ∂U and with $u_0 \in C^0(\overline{U})$. Complete the following to show $u \equiv v$.

(a) Consider the difference $w = u - v$. Find an initial/boundary value problem satisfied for w.

(b) Consider the square

$$A(t) = \int_U w^2$$

of the L^2 norm of w, and show $A'(t) \leq 0$. Hint: Differentiate under the integral sign, use the equation, and apply the divergence theorem. Hint hint: Show

$$\text{div}(wDw) = |Dw|^2 + w\Delta w.$$

(c) Conclude $w \equiv 0$. Hint: The IVP $A' = 0$, $A(0) = 0$ has a unique solution.

Problem 5 (mollification) In Assignment 3 Problem 7 the mollification of a function $u \in L^1_{loc}(\mathbb{R}^n)$ is considered. The objective of this problem is to consider a little more carefully the mollification of a function defined on a bounded open set $U \subset \mathbb{R}^n$ for application to harmonic functions.

Let $u \in C^0(U)$ and recall the mollification formula

$$u * \phi_\delta(\mathbf{p}) = \int_{\mathbf{x} \in \mathbb{R}^n} u(\mathbf{x}) \; \phi_\delta(\mathbf{p} - \mathbf{x}). \tag{1}$$

which is intended to associate with u some other smooth function, or more properly a family of smooth functions indexed by a positive parameter δ. Here we take explicitly

$$\phi_\delta(\mathbf{x}) = \frac{1}{\delta^n} \phi_1\left(\frac{\mathbf{x}}{\delta}\right)$$

and

$$\phi_1(\mathbf{x}) = \begin{cases} ce^{-\frac{1}{1-|\mathbf{x}|^2}}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \geq 1 \end{cases}$$

where the positive constant c is chosen so that $\int_{\mathbb{R}^n} \phi_1 = \int_{\mathbb{R}^n} \phi_\delta = 1$.

Finally, given $\mathbf{p} \in U$ set

$$R = \operatorname{dist}(\mathbf{p}, \partial U) = \max_{\mathbf{x} \in \partial U} |\mathbf{x} - \mathbf{p}|$$

and

$$U_R = \{\mathbf{x} \in U : \operatorname{dist}(\mathbf{x}, \partial U) < R\}.$$

(a) Explain why the mollification formula (1) does not determine a well-defined value if $\delta > R$.

(b) Give an example showing the mollification formula (1) may not determine a well-defined finite value if $\delta = R$.

(c) Explain why the mollification formula (1) can and does make sense for $\delta < R$.

(d) Show U_R is an open set and $u * \phi_\delta \in C^\infty(U_R)$ for $\delta < R$. Hint(s): Recall that

$$C^\infty(U_R) = \bigcap_{k=1}^{\infty} C^k(U_R).$$

An easy way to show $u * \phi_\delta \in C^k(U_R)$ is to show every derivative

$$D^\beta(u * \phi_\delta) \qquad \text{for} \qquad |\beta| = k + 1$$

of order $k + 1$ is well-defined.

Problem 6 (regularity for solutions of Laplace's equation) Let U be an open subset of $\mathbb{R}^2$. Complete the following steps to show that if $u \in C^2(U)$ satisfies $\Delta u = 0$, then $u \in C^\infty(U)$.

(a) Fix $\mathbf{p} \in U$ and take $\delta < \text{dist}(\mathbf{p}, \partial U)$. Show that for some $\epsilon > 0$ the mollification $u * \phi_\delta$ is well-defined and satisfies

$$u * \phi_\delta \in C^\infty(B_\epsilon(\mathbf{p})).$$

Pay careful attention to part **(c)** of Problem 5 above.

(b) Write down the formula for $u * \phi_\delta(\mathbf{x})$ for $\mathbf{x} \in B_\epsilon(\mathbf{p})$ and show

$$u * \phi_\delta(\mathbf{x}) = \int_0^\delta \left(\int_{\mathbf{q} \in \partial B_r(\mathbf{x})} u(\mathbf{q}) \ \phi_\delta(\mathbf{x} - \mathbf{q}) \right) dr$$

Hint: Consider the integral over $B_\delta(\mathbf{x})$ as a limit of Riemann sums with partition pieces that also partition spherical shells of radius (approximately) r_j and thickness $r_{j+1} - r_j$. This is sometimes called integration using **generalized spherical coordinates**.

(c) Show that for $\mathbf{q} \in \partial B_r(\mathbf{x})$ the value

$$\phi_\delta(\mathbf{x} - \mathbf{q}) = \frac{1}{\delta^n} \phi_1 \left(\frac{\mathbf{x} - \mathbf{q}}{\delta} \right)$$

is independent of $\mathbf{q}$ but only depends on r. Call this value $\mu(r)$.

(d) Show

$$u * \phi_\delta(\mathbf{x}) = n\omega_n \ u(\mathbf{x}) \int_0^\delta r^{n-1} \mu(r) \, dr$$

Hint: Use the mean value property.

(e) Show

$$n\omega_n \int_0^\delta r^{n-1} mu(r) \, dr = 1$$

Hint(s): Express $n\omega_n r^{n-1} \mu(r)$ as an integral over $\partial B_r(\mathbf{x})$, and change variables back to rectangular coordinates from generalized spherical coordinates.

(f) Explain why this implies $u \in C^\infty(U)$.

Problem 7 (uniqueness of the Dirichlet problem for Poisson's equation) In Problem 2 and again in Problem 4 of Assignment 6 proofs of uniqueness of solutions for the Dirichlet boundary value problem

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g \end{cases} \tag{2}$$

are given/outlined. Recall that the functions f and g were assumed continuous on their respective domains. Were these assumptions of continuity used in the proofs? If so, explain where the proof breaks down. If not, state a stronger uniqueness assertion which applies to some class of disontinuous functions.

Problem 8 (fundamental solution for the 1-D heat equation; Haberman 10.4) We have considered special solutions of the heat equation having the form

$$u(x,t) = e^{-j^2\pi^2 t/L^2} \cos\left(\frac{j\pi}{L}x\right) \qquad \text{and} \qquad u(x,t) = e^{-j^2\pi^2 t/L^2} \sin\left(\frac{j\pi}{L}x\right)$$

on the interval $[0, L]$. These are **separated variables** solutions. They can, of course, also be considered as solutions on all of the spatial domain $\mathbb{R}$, but that consideration is not so interesting because they are spatially periodic. There is another important solution of the heat equation to know about and remember.

The function $\Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ given by

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$$

is called the **fundamental solution** of the one-dimensional heat equation. Notice that the fundamental solution does not have the form of a separated variables solution.

(a) Verify that Φ satisfies $u_t = u_{xx}$ for $(x,t) \in \mathbb{R} \times (0, \infty)$.

(b) Use L'Hopital's rule to determine

$$\lim_{t \searrow 0} \Phi(x,t).$$

(c) Make an animation of the spatial graph of the fundamental solution Φ with animation parameter t.

Problem 9 (fundamental solution for the 1-D heat equation; Haberman 10.4) Calculate the spatial L^1 norm

$$I(t) = \int_{x\in\mathbb{R}} \Phi(x,t)$$

of the fundamental solution. Hint(s): Note that $I(t) = 2J(t)$ where

$$J(t) = \int_0^\infty \Phi(x,t)\,dx.$$

Calculate $J(t)^2$. Use y as a spatial variable of integration in one of the factors $J(t)$. Write what you get as an iterated integral and then as an integral of a function of two variables over the first quadrant. Use polar coordinates.

Problem 10 (fundamental solution for the 1-D heat equation; Haberman 10.4)

(a) How could you modify Φ so that it satisfies $u_t = ku_{xx}$ for non-unitary conductivity? Hint(s): Consider **scaling** in the spatial variable and/or the time variable. and time. See how Haberman defines as the fundamental solution.

(b) How can you modify the one-dimensional fundamental solution of the heat equation to obtain the fundamental solution of the heat equation $\Phi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ on (all of) $\mathbb{R}^n$?