# Assignment 7:
# Wave equation and Sturm Liouville Theory
# Due Tuesday December 7, 2021

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**Problem 1** *(d'Alembert's solution, Haberman 4.4.6-8) Consider the initial value problem for the 1-D wave equation*

$$\begin{cases} u_{tt} = \sigma^2 u_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = v_0(x), & x \in \mathbb{R} \end{cases}$$

*where $u_0, v_0 \in C^1(\mathbb{R})$ are two given functions.*

**(a)** *Let $Lu = u_t + \sigma u_x$ and $Mu = u_t - \sigma u_x$. Use the method of characteristics (twice) applied to the factorization $\Box u = MLu = 0$ to obtain d'Alembert's solution*

$$u(x,t) = \frac{1}{2}[u_0(x - \sigma t) + u_0(x + \sigma t)] + \frac{1}{2\sigma} \int_{x-\sigma t}^{x+\sigma t} v_0(\xi)\, d\xi.$$

**(b)** *Show that if $u_0$ and $v_0$ are both odd and periodic with period $2L$, then the restriction of d'Alembert's solution to $[0, L] \times (0, \infty)$ satisfies the initial/boundary value problem*

$$\begin{cases} u_{tt} = \sigma^2 u_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ u(0,t) = 0 = u(L,t), & t \geq 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = v_0(x), & x \in \mathbb{R} \end{cases}$$

**(c)** *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} = \sigma^2 u_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ u(0,t) = 0 = u(L,t), & t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R} \\ u_t(x,0) = h(x), & x \in \mathbb{R} \end{cases}$$

*where $g, h \in C^1[0, L]$. Can d'Alembert's solution be applied to solve this problem? Why or why not?*

**Problem 2** *(The heat equation on $\mathbb{R}$; fundamental solution, Haberman section 10.4)* *We have considered special solutions of the heat equation having the form*

$$u(x,t) = e^{-j^2\pi^2 t/L^2} \cos\left(\frac{j\pi}{L}\, x\right) \qquad \text{and} \qquad u(x,t) = e^{-j^2\pi^2 t/L^2} \sin\left(\frac{j\pi}{L}\, x\right)$$

*on the interval $[0, L]$. These are* **separated variables** *solutions. They can, of course, also be considered as solutions on all of the spatial domain $\mathbb{R}$, but that consideration is not so interesting because they are spatially periodic. There is another important solution of the heat equation to know about and remember.*

*The function $\Phi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ given by*

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$$

*is called the* **fundamental solution** *of the one-dimensional heat equation. Notice that the fundamental solution does not have the form of a separated variables solution.*

**(a)** *Verify that $\Phi$ satisfies $u_t = u_{xx}$ for $(x,t) \in \mathbb{R} \times (0, \infty)$.*

**(b)** *Use L'Hopital's rule to determine*

$$\lim_{t \searrow 0} \Phi(x,t).$$

**(c)** *Make an animation of the spatial graph of the fundamental solution $\Phi$ with animation parameter $t$.*

**(d)** *Calculate the spatial $L^1$ norm*

$$I(t) = \int_{x \in \mathbb{R}} \Phi(x,t)$$

*of the fundamental solution. Hint(s): Note that* $I(t) = 2J(t)$ *where*

$$J(t) = \int_0^\infty \Phi(x,t)\,dx.$$

*Calculate* $J(t)^2$*. Use* $y$ *as a spatial variable of integration in one of the factors* $J(t)$*. Write what you get as an iterated integral and then as an integral of a function of two variables over the first quadrant. Use polar coordinates.*

**(e)** *How could you modify* $\Phi$ *so that it satisfies* $u_t = k u_{xx}$ *for non-unitary conductivity? Hint(s): Remember Problem 1 from Assignment 3 = Exam 1 about scaling in space and time. Also read section 10.4 of Haberman and see what Haberman defines as the fundamental solution.*

**(f)** *Given* $u_0 : \mathbb{R} \to \mathbb{R}$ *with* $u \in C^0(\mathbb{R})$*, the function*

$$u(x,t) = \int_{\xi\in\mathbb{R}} \Phi(x-\xi, t)\ u_0(\xi)$$

*is called the* **spatial convolution** *of the fundamental solution with* $u_0$*. Show that this spatial convolution satisfies the initial value problem*

$$\begin{cases} u_t = u_{xx} & \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

*for the heat equation on the whole real line.*

**(g)** *(Bonus) How can you modify the one-dimensional fundamental solution of the heat equation to obtain the fundamental solution of the heat equation* $\Phi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ *on (all of)* $\mathbb{R}^n$*?*

**Problem 3** *(wave equation: transverse oscillation model; Haberman section 4.2)*

*In the transverse oscillation model for the wave equation (think of the wave equation as a model for the vibration of a guitar string) one may assume a tension* $T$ *and a lineal density* $\rho$ *for the string. Thus, the mass of a small portion of the string modeled by the graph of a function* $u : [0,L] \to \mathbb{R}$ *over the interval* $[a,b] \subset (0,L)$ *is approximately* $\rho(b-a)$*. Thus, Newton's second law of motion (*$F = Ma$*) for the* **transverse displacement** $u$ *gives (approximately)*

$$Ma_{\text{transverse}} = \rho(b-a)\ u_{tt}(x^*,t) \approx F_{\text{transverse}}(b,t) + F_{\text{transverse}}(a,t) \tag{1}$$

*where* $x^*$ *is some point with* $x^* \in (a,b)$ *and* $F_{\text{transverse}}$ *is the component of the tension force orthogonal to the (equilibrium of the) string, i.e., the vertical component if the equilibrium of your string is horizontal.*

**(a)** *Draw a picture of the graph of $u$, representing the position of the string above the interval $(a, b)$ at time $t$. Draw your picture so that $u_x(a, t)$ and $u_x(b, t)$ are different, i.e., so that the string is curving over the interval $[a, b]$.*

**(b)** *If $T$ represents the tension in the string to the right and is always tangent to the string, use similar triangles to find the components $F_{\text{transverse}}(a, t)$ and $F_{\text{transverse}}(b, t)$ of $T$ at $x = a$ and $x = b$ respectively.*

**(c)** *Substitute your values from part (b) into (1), divide by $b - a$ and take the limit as $b, a \to x$ to obtain the wave equation for $u$.*

**Problem 4** *(variable tension, transverse oscillations of a hanging chain; O'Neil Advanced Engineering Mathematics section 6.1)*

**(a)** *If your derivation in Problem 3 assumed the tension $T$ was constant, go back and derive the wave equation for transverse oscillations in which $T = T(x)$ depends on the position $x$.*

**(b)** *Use your model to find the fundamental modes of oscillation (separated variables solutions) for a hanging chain.*

**(c)** *Animate some of your fundamental modes.*

**Problem 5** *(transverse oscillations of a rectangular drum, Haberman section 7.3)*

**(a)** *Find the separated variables solutions of the boundary value problem*

$$\begin{cases} u_{tt} = \Delta u, & \text{on } R \times [0, \infty) \\ u\big|_{\partial R} \equiv 0, & t > 0 \end{cases}$$

*for the wave equation on the rectangle $R = [0, L] \times [0, M]$.*

**(b)** *Choose specific positive values for L and $M$, and animate some of the fundamental modes you found in part (a).*