Assignment 6: Fourier series and Laplace's equation
Due Wednesday March 18, 2026

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In the following problems, let L and M denote positive real numbers.

The Laplacian of a function $u = u(x, y)$ of two variables is given by the sum of the homogeneous second partial derivatives:

$$\Delta u = u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

A solution of Laplace's equation $\Delta u = 0$ is called **harmonic**.

Problem 1 (Haberman Exercise 2.5.1(a)) Use separation of variables to solve the boundary value problem for Laplace's equation on the rectangular domain $[0, L] \times [0, M]$:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on} \quad (0, L) \times (0, M) \\ u_x(0, y) = 0 = u_x(L, y), & 0 < y < M \\ u(x, 0) = 0, & 0 < x < L \\ u(x, M) = x^3 - 3Lx^2/2 + L^3/2, & 0 < x < L. \end{cases} \tag{1}$$

Problem 2 (Haberman Exercise 2.5.1(a)) Use mathematical software to plot your solution from Problem 1 above for some interesting choice of the side lengths L and M.

Problem 3 (Haberman Exercise 7.2.3(a) and (c)) Use separation of variables (twice) to solve the initial/bounary value problem for the heat equation on the rectangular domain $R = (0, L) \times (0, M)$:

$$\begin{cases} u_t = u_{xx} + u_{yy} & \text{on} \quad (0, L) \times (0, M) \times (0, \infty) \\ u_x(0, y) = 0 = u_x(L, y), & 0 < y < M \\ u(x, 0) = 0, & 0 < x < L \\ u(x, M) = x^3 - 3Lx^2/2 + L^3/2, & 0 < x < L \\ u(x, y, 0) = u_0(x, y), & (x, y) \in (0, L) \times (0, M). \end{cases} \tag{2}$$

Suggestion/hint: Find the initial/boundary value problem satisfied by $v = u - w$ where w is the Fourier series solution you found in Problem 1. Solve for v and then give your final answer as $u = v + w$.

Problem 4 (Haberman Exercise 8.2.4) You should have a formula/Fourier series for the solution of Problem 3 depending on the initial temperature distribution u_0. Find

$$\lim_{t\to\infty} u(x, y, t). \tag{3}$$

Problem 5 (Haberman section 2.5.2 and Exercise 2.5.4) Given $a > 0$, let the domain Ω be given by

$$\Omega = B_a(\mathbf{0}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}. \tag{4}$$

Such a domain is called a "ball" or a disk (of radius a and with center $\mathbf{0}$) in two dimensions.

(a) The **boundary** $\partial\Omega$ of the disk Ω is a circle. Express the set $\partial\Omega = \partial B_a(\mathbf{0})$ in "set builder" notation as in (4).

(b) The **polar coordinates map** $\psi : R \to \Omega$ is given by $\psi(r, \theta) = (r\cos\theta, r\sin\theta)$.

(i) If R is the rectangular domain

$$R = (0, a) \times (0, 2\pi)$$

find the continuous extension of ψ to ∂R.

(ii) Notice $\psi(R) = \{\psi(r, \theta) : (r, \theta) \in R\} \neq \Omega$. Identify the range $W = \psi(R)$ precisely.

(iii) $\psi : R \to W$ has an inverse. Discuss the possibility of extending ψ^{-1} to the closure of W which is

$$\overline{W} = \overline{B_r(\mathbf{0})} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2\}.$$

(c) Consider the problem

$$\begin{cases} \Delta u = 0, & (x,y) \in \Omega \\ u\big|_{\partial\Omega} = u_0 \end{cases}$$

where $u_0 = u_0(x,y) : \partial\Omega \to \mathbb{R}$ is a given function, and consider $v : R \to \mathbb{R}$ by $v(r,\theta) = u \circ \psi(r,\theta)$.

(i) Use the chain rule to compute

$$Dv = \left(\frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}\right) = (v_r, v_\theta)$$

and

$$D^2 v = \begin{pmatrix} v_{rr} & v_{r\theta} \\ & \\ v_{\theta r} & v_{\theta\theta} \end{pmatrix}$$

in terms of Du and D^2u. Hint:

$$\frac{\partial v}{\partial r} = u_x \circ \psi \ \cos\theta + u_x \circ \psi \ \sin\theta.$$

(ii) Using your calculations from part **(b)(i)** above and the PDE $\Delta u = 0$, find a partial differential equation satisfied by v on R.

(iii) Formulate some appropriate boundary conditions for $v : R \to \mathbb{R}$ on ∂R.

(d) Formulate and use separated variables to solve an appropriate boundary value problem for v on R.

(e) Find

$$u(0,0) = \lim_{r \to 0} v(r,\theta).$$

Problem 6 (Haberman Exercise 2.5.5(d)) Let $Q = \{(x,y) \in B_a(\mathbf{0}) : x, y > 0\}$ be the quarter disk in the first quadrant. Consider the boundary value problem

$$\begin{cases} \Delta u = 0, \text{on} \quad Q & \\ u_y(x,0) = 0, & 0 < x < a \\ (x,y) \cdot Du(x,y) = f(x,y), & x^2 + y^2 = a^2 \\ u_x(0,y) = 0, & 0 < y < a. \end{cases} \tag{5}$$

(a) Notice the Laplacian is given by a divergence, and use the divergence theorem to derive a condition that must be satisfied by the flux $f : \partial Q \cap \partial B_a(\mathbf{0}) \to \mathbb{R}$ (if there exists a solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of (5)).

(b) Assuming the condition you derived in part **(a)** does hold for the prescribed flux f, use polar coordinates and separation of variables to find a solution u.

(c) Take $a = 3$, pick an interesting specific choice for f, and use mathematical software to plot (some partial sums of) your solution.

Problem 7 (mean value property; Haberman section 2.5.4) Let $a > 0$ and $p = (p_1, p_2)$ be a point in $\mathbb{R}^2$. The disk with radius a and center p is the domain

$$B_a(p) = \{(x,y) \in \mathbb{R}^2 : (x - p_1)^2 + (y - p_2)^2 < a^2\}.$$

(a) Solve the problem

$$\begin{cases} \Delta u = 0, \text{on} \quad B_a(p) & \\ u(x,y) = u_a(x,y), & (x-p_1)^2 + (y-p_2)^2 = a^2 \end{cases}$$

where $u_a : \partial B_a(p) \to \mathbb{R}$ is a given function. Hint: $w(x,y) = u(x + p_1, y + p_2)$ defines a solution of Laplace's equation on $B_a(\mathbf{0})$. Quote/use your solution from Problem 5 above.

(b) Assume $\Delta u = 0$ on some domain Ω with

$$\overline{B_a(p)} \subset \Omega.$$

Show

$$u(p) = \frac{1}{2\pi a} \int_{\partial B_a(p)} u. \tag{6}$$

Hint: To integrate over a circle, parameterize the circle by $\gamma(t) = p + a(\cos t, \sin t)$ and then integrate the composition against the arcspeed of the parameterization:

$$\int_0^{2\pi} u \circ \gamma(t) \; |\gamma'(t)| \, dt.$$

The formula (6) is called the **mean value property** for harmonic functions. It says that when the closure of a disk is contained in a domain, then the value of a harmonic function on that domain is given by the average value on the boundary of the disk.

Problem 8 (maximum principle, Haberman section 2.5.4) Let Ω be a bounded domain in $\mathbb{R}^2$. This means there is some $R > 0$ so that $\Omega \subset B_R(\mathbf{0})$. Assume also that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ so that

$$M_{\partial\Omega} = \max_{(x,y)\in\partial\Omega} u(x,y) < \infty.$$

(a) (weak maximum principle) Show that if $(x, y) \in \Omega$, then $u(x, y) \leq M$. Hint: Assume $u(x, y) > M$ for some $(x, y) \in \Omega$, then for some $p \in \Omega$ there holds

$$u(p) = M_{\overline{\Omega}} = \max_{(x,y)\in\overline{\Omega}} u(x,y) > M_{\partial\Omega}. \tag{7}$$

Hint hint: Let $R_p = \max\{r : B_r(p) \subset \Omega\}$. Show there is some $q \in \partial B_{R_p}(p) \cap \partial\Omega$ and

$$u(q) = \lim_{a \nearrow R_p} u(p + a(q - p)/R_p) = u(p).$$

(This contradicts the assumption (7).) Hint hint hint: Use the mean value property from Problem 7 to show $u(p + a(q - p)/R_p) = u(p)$ for $0 < a < R_p$.

(b) (strong maximum principle) Show that if the domain Ω is also **connected**, then the reasoning in part **(a)** leads to a much stronger conclusion: Either

$$u(x, y) < M_{\partial\Omega} \qquad \text{for every } (x, y) \in \Omega,$$

or u is a constant function.

Connected means that given any two points p and q in Ω there is a continuous path in Ω connecting p and q. A continuous path can be parameterized by a continuous function $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Hint(s): Repeat the reasoning of part **(a)** under the assumption $u(p) = M_{\partial\Omega}$ for some $p \in \Omega$. Conclude u is constant on $B_{R_p}(p)$. Next assume there is some $q \in \Omega$ with $u(q) < u(p)$; take a path connecting p and q and derive a contradiction.

Problem 9 (uniqueness of solutions for the boundary value problem for Poisson's equation, Haberman Exercise 2.5.10) Let $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be two solutions of the boundary value problem

$$\begin{cases} \Delta u = f, & \text{on} \quad \Omega \\ u\big|_{\partial\Omega} = g, & \text{on} \quad \partial\Omega \end{cases} \tag{8}$$

for Poisson's equation $\Delta u = f$ where $f \in C^0(\Omega)$ is a given continuous function and $g : \partial\Omega \to \mathbb{R}$ is any function. Show $u_1 \equiv u_2$, that is, solutions of this problem are unique.

Problem 10 Show the following functions are solutions of Laplace's equation:

(a) Let $u = u(x, y)$ be the real part of $(x + iy)^3$.

(b) Let $u = u(x, y)$ be the imaginary part of e^{x+iy}.

(c) Let $u = u(x, y)$ be a function for which along with another function $v = v(x, y)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note: The Cauchy-Riemann equations are a system of two first order partial differential equations.