Assignment 6: Laplace's Equation

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Problem 1 (weak maximum principle) Let U be a bounded open subset of $\mathbb{R}^n$, and let $u \in C^2(U) \cap C^0(\overline{U})$ be a harmonic function, i.e., a solution of Laplace's equation

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Simply by the continuity of u on the closure of the bounded set U, the number

$$M = \max_{\mathbf{x} \in \overline{U}} u$$

is a well-defined finite real number. Similarly, ∂U is a closed and bounded set on which u is continuous, so the maximum

$$\max_{\mathbf{x} \in \partial U} u$$

of u on ∂U is also a well-defined finite number.

The weak maximum principle asserts that

$$M = \max_{\mathbf{x} \in \partial U} u, \tag{1}$$

that is, the global maximum of a harmonic function u is taken on the boundary. In particular, there is some $\mathbf{p} \in \partial U$ for which $u(\mathbf{p}) = M$.

Assume by way of contradiction that there is some point $\mathbf{q} \in U$ with

$$u(\mathbf{q}) > m = \max_{\mathbf{x} \in \partial U} u, \tag{2}$$

and complete the following steps to prove the weak maximum principle:

(a) For $\epsilon > 0$, consider $v : \mathbb{R}^n \to \mathbb{R}$ by

$$v(\mathbf{x}) = u(\mathbf{q}) - \epsilon|\mathbf{x} - \mathbf{q}|^2.$$

Show that if ϵ is small enough, then

$$v(\mathbf{x}) > m \qquad \text{for all } \mathbf{x} \in \overline{U}.$$

(b) Fix ϵ satisfying the condition of part **(a)**. For $\delta > 0$ consider $w : \mathbb{R}^n \to \mathbb{R}$ by

$$w(\mathbf{x}) = v(\mathbf{x}) + \delta.$$

Show that for some $\delta \geq 0$, the function w satisfies the following:

(i) $w(\mathbf{x}) \geq u(\mathbf{x})$ for $\mathbf{x} \in \overline{U}$,

(ii) $w(\mathbf{x}) > m$ for all $\mathbf{x} \in \overline{U}$.

(iii) $w(\mathbf{y}) = u(\mathbf{y})$ for some $\mathbf{y} \in U$.

Hint: Consider the number $a = \max\{u(\mathbf{x}) - v(\mathbf{x}) : \mathbf{x} \in \overline{U}\}$.

(c) Show that it follows from conditions **(i)** and **(ii)** of part **(b)** above that $\Delta u(\mathbf{y}) < 0$. Hint(s): You may want to consider the low dimensional cases $n = 1$ and $n = 2$ first. Show for each $j = 1, 2, \ldots, n$

$$\frac{\partial^2 u}{\partial x_j}(\mathbf{y}) \leq \frac{\partial^2 w}{\partial x_j}(\mathbf{y}).$$

Compute Δw.

(d) Note that what you have shown in part **(c)** constitutes a contradiction and conclude there is no point $\mathbf{q} \in U$ for which (2) holds. Explain why this implies the weak maximum principle (1).

Problem 2 (uniqueness of solutions for Poisson's equation) Assume U is a bounded open subset of $\mathbb{R}^n$ and f and g are functions satisfying

(i) $f \in C^0(U)$ and

(ii) $g \in C^0(\partial U)$.

Given $u, v \in C^2(U) \cap C^0(\partial U)$ satisfing the Dirichlet boundary value problem(s)

$$\begin{cases} \Delta u = f, & \mathbf{x} \in U \\ u\big|_{\partial U} = g & \mathbf{x} \in \partial U \end{cases} \tag{3}$$

and

$$\begin{cases} \Delta v = f, & \mathbf{x} \in U \\ v\big|_{\partial U} = g & \mathbf{x} \in \partial U, \end{cases}$$

use the weak maximum principle of Problem 1 above to show $u \equiv v$. Hint: Find the boundary value problem satisfied by $w = u - v$.

Problem 3 (Laplace's equation and the heat equation) Let $U = [0, 1] \times [0, 1]$ be the unit square in the first quadrant, and let $u_0 = u_0(x, y)$ be any function satisfying

(i) $u_0(x, y) = x^2 - y^2$ for $(x, y) \in \partial U$,

(ii) $u_0 \in C_0(\overline{U})$.

(a) Solve the initial/boundary value problem

$$\begin{cases} u_t = u_{xx} + u_{yy}, & (x, y, t) \in U \times (0, \infty) \\ u(x, y, t) = x^2 - y^2, & (x, y) \in \partial U, \ t > 0 \\ u(x, y, 0) = u_0(x, y), & (x, y) \in U. \end{cases}$$

Hint: Solve for $v(x, y) = u(x, y) - x^2 + y^2$ using a Fourier basis involving the functions $w_{jk} = \sin(j\pi x)\ \sin(k\pi y)$ for $j, k = 1, 2, 3, \ldots$.

(b) Using your solution from part **(a)** compute

$$\lim_{t \nearrow \infty} u(x, y, t).$$

Problem 4 (Haberman 2.5.12) In Problem 2 above the weak maximum principle is used to show uniqueness of solutions for the Dirichlet boundary value problem for Poisson's equation (3). Complete the following steps to give a second proof of this result.

(a) Use the coordinate expression

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$$

for the divergence of a vector field $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ defined on a region $\mathcal{U} \subset \mathbb{R}^n$ to derive the product formula

$$\operatorname{div}(\phi \mathbf{v}) = D\phi \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v}$$

for the scaled field $\phi \mathbf{v}$ where $\phi : \mathcal{U} \to \mathbb{R}$ is a scalar function.

(b) Obtain an identity for

$$\int_{\mathcal{U}} w \Delta w.$$

Hint(s): Use part (a) and remember $\Delta w = \operatorname{div} Dw$.

(c) Prove the boundary value problem

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial \mathcal{U}} = g \end{cases} \tag{4}$$

for Poisson's equation has a unique solution. Hint(s): Note that your identity in (b) holds for any function. Take $w = u - v$ where u and v are two solutions of (4).

Problem 5 (mean value property) Consider

$$f(r) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{p})} u$$

where $B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{p}| < r\}$ and $u : \mathcal{U} \to \mathbb{R}$ is a solution of Laplace's equation with $\overline{B_r(\mathbf{p})} \subset \mathcal{U} \subset \mathbb{R}^2$.

(a) Compute $f'(r)$ and show $f'(r) = 0$. Hint(s): Change variables so that you're integrating on the boundary of a fixed ball of radius 1. Differentiate under the integral sign, and use the divergence theorem.

(b) Use continuity to conclude

$$u(\mathbf{p}) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{p})} u.$$

(c) (Bonus) Show

$$u(\mathbf{p}) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{p})} u.$$

Hint(s): Parameterize $B_r(\mathbf{p}) \backslash \{(0,0)\}$ on the cylinder $\mathbb{S}^1 \times (0, r)$ where $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$ is the unit circle. Use Fubini's theorem/iterated integrals on the cylinder and part **(b)** to replace the value of the inner integral on each circle.

Problem 6 (stong maximum principle) Consider the assertion (1) of the weak maximum principle. Note that while this does say there exists some $\mathbf{p} \in \partial U$ with

$$u(\mathbf{p}) = M = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x})$$

it does not rule out the possibility that there exists some $\mathbf{q} \in U$ with

$$u(\mathbf{q}) = M = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x}). \tag{5}$$

In fact this can happen, for example, if $u \equiv c$ is constant. The **strong maximum principle** says this is essentially the only way the equality in (5) is possible:

Theorem 1 (E. Hopf strong maximum principle) Under the assumptions of the weak maximum principle, namely if U is a bounded, open, and **connected** subset of $\mathbb{R}^n$ and $u \in C^2(U) \cap C^0(\overline{U})$ is harmonic, then either

$$u(\mathbf{q}) < M = \max_{\mathbf{x}\in\overline{U}} u(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in U$$

or $u \equiv c$ is constant.

Assume (5) holds for some $\mathbf{q} \in U$, and complete the following steps to prove the strong maximum priciple using the mean value property of Problem 7 part **(b)**:

(a) Show $u(\mathbf{x}) \equiv u(\mathbf{q})$ for $\mathbf{x} \in \partial B_r(\mathbf{q})$ whenever $\overline{B_r(\mathbf{q})} \subset U$. Hint(s): Assume by way of contradition that $u(\mathbf{p}) < u(\mathbf{q})$ for some $\mathbf{p} \in \partial B_r(\mathbf{q})$. Use continuity to show

$$\int_{\partial B_r(\mathbf{q})} u < u(\mathbf{q}).$$

(This contradicts the mean value property.)

Use the following definition of what it means for U to be connected:

Definition 1 (connected) An open set $U \subset \mathbb{R}^n$ is **connected** if given any two points $\mathbf{q}$ and $\mathbf{x}$ in U, there exists a continuous function $\alpha : [0,1] \to U$ with $\alpha(0) = \mathbf{q}$ and $\alpha(1) = \mathbf{x}$. The function α is called a **path** connecting $\mathbf{q}$ to $\mathbf{x}$ in U.

(b) Show $\max\{t \in [0,1] : u(\alpha(\tau)) = u(\mathbf{q}) \text{ for } 0 \le \tau \le t\} = 1$.

(c) Explain how the steps **(a)** and **(b)** above constitute a proof of the strong maximum principle.

Problem 7 (Laplace's equation) Find all separated variables solutions $u(x,y) = A(x)B(y)$ of the boundary value problem

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,L) \times (0,M) \\ u(x,0) = 0 = u(x,M), & x \in (0,L) \end{cases}$$

where $L, M > 0$.

Problem 8 (Homogeneous boundary conditions on a rectangle)

(a) Solve the boundary value problem for Laplace's equation

$$\begin{cases} \Delta u = 0, & (x,y) \in (0,2) \times (0,\pi) \\ u(x,0) = 0 = u(x,\pi), & x \in (0,2) \\ u(0,y) = 0, & y \in (0,\pi) \\ u(2,y) = \sin y, & y \in (0,\pi). \end{cases}$$

(b) Use mathematical software to plot your solution.

Problem 9 (Haberman 2.5.1) Let $R = (0,L) \times (0,M)$ be a fixed rectangle in the plane modeling a heat conducting plate. Solve the boundary value problem for Laplace's equation (equilibrium solution of the heat equation):

$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = Lx - x^2, & 0 < x < L \\ u(x,M) = 0, & 0 < x < L \\ u(0,y) = 0, & 0 < y < M \\ u(L,y) = 0, & 0 < y < M. \end{cases} \tag{6}$$

Hint: Initially set aside the boundary condition associated with $y = 0$ and find all separated variables solutions $u(x,y) = A(x)B(y)$. Then use a superposition of these solutions.

Problem 10 (Laplace's equation in a strip, Haberman 2.5.15) Solve the boundary value problem for Laplace's equation

$$\begin{cases} \Delta u = 0 & \text{on } (0,L) \times (0,\infty) \\ u(0,y) = 0 = u(L,y), & y > 0 \\ u(x,0) = g(x), & 0 < x < L \end{cases}$$