Assignment 5: Heat Equation (part II)
Due Wednesday March 4, 2026

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In the following problems, let L denote a positive real number.

Problem 1 (Haberman 1.4.4) Assume heat conduction is modeled in a thin metal rod by

$$u_t = (ku_x)_x \qquad \text{on} \qquad (0, L) \times (0, \infty)$$

where $k = k(x)$ depends on position. If both ends of the rod are modeled as insulated, show the total heat energy in the rod must be constant (as a function of time).

Problem 2 (Haberman 1.4.5) Assume heat conduction is modeled in a thin metal rod by

$$u_t = ku_{xx} \qquad \text{on} \qquad (0, L) \times (0, \infty)$$

where the diffusivity k is a constant. Give an expression for the temperature $U(L)$ of an equilibrium solution $U = U(x)$ with $U(0) = T_1$ and $U_x(0) = r_1$.

Problem 3 (Haberman 1.4.6) If heat conduction in a thin metal rod is modeled by the forced 1-D heat equation with nonzero constant source term Q, and both ends are modeled as insulated, prove there can be no equilibrium solution

$$U(x) = \lim_{t \nearrow \infty} u(x, t).$$

Problem 4 Use Fourier's law to determine an appropriate boundary condition on an n-dimensional region R corresponding to heat conduction in R with **insulated boundary**.

Problem 5 Solve the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < L,\ t > 0 \\ u(0,t) = 0 = u(L,t), & t > 0 \\ u(x,0) = \sin(3\pi x/L), & 0 \le x \le L. \end{cases}$$

Hint: Look for a solution of the form $u(x,t) = f(t)\sin(3\pi x/L)$.

Problem 6 Take $L = 1$ and

(a) Use numerical software to plot your solution to Problem 5 as the graph of a function of two variables (in three dimensions).

(b) Use numerical software to animate your solution to Problem 5 (with time as the animation parameter).

Problem 7 Solve the initial/boundary value problem

$$\begin{cases} u_t = k\Delta u, & (x,y,t) \in R \times (0,\infty) \\ u_x(0,y,t) = 0 = u_x(L,y,t), & t > 0,\ 0 < y < M \\ u_y(x,0,t) = 0 = u_y(x,M,t), & t > 0,\ 0 < x < L \\ u(x,y,0) = g(x,y), & (x,y) \in R \end{cases} \tag{1}$$

for the 2-D heat equation on the rectangular region

$$R = (0,L) \times (0,M) = \{(x,y) : 0 < x < L,\ 0 < y < M\}$$

where the initial temperature distribution is given by

$$g(x,y) = 1 + \cos\left(\frac{2\pi}{L}\,x\right)\cos\left(\frac{3\pi}{M}\,y\right).$$

Problem 8 Take $k = 1$, $L = 1$, and $M = 2$ and

(a) Use numerical software to plot $u(x,1,t)$ as the graph of a function of two variables using your solution to Problem 7.

(b) Explain the meaning of the plot in part **(a)**.

(c) Use numerical software to animate your solution to Problem 7 (with time as the animation parameter).

Problem 9 (Haberman 2.4.1 part (a)) Solve the boundary value problem for the 1-D heat equation:

$$\begin{cases} u_t = k u_{xx}, & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = 0, & t > 0 \\ u_x(L,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 < x < L \end{cases} \tag{2}$$

where

$$g(x) = \begin{cases} 0, & x < L/2 \\ 1/2, & x = L/2 \\ 1, & x > L/2. \end{cases}$$

Problem 10 Take $k = 3$ and $L = 2$ and

(a) Use numerical software to plot $u(x,t)$ as the graph of a function of two variables using your solution to Problem 9.

(b) Use numerical software to animate your solution to Problem 9 (with time as the animation parameter).