Assignment 5: Heat Equation

Pace: Thursday October 10, 2024, Due Tuesday October 15, 2024

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Problem 1 (Haberman 1.2.8) Give an expression for the total thermal energy in a rod modeled on an interval $0 \leq x \leq \ell$ in terms of the temperature $u = u(x,t)$.

Problem 2 If f is a continuous function defined on the interval $[0,\ell]$ with $\ell > 0$ and

$$\int_a^b f(x)\,dx = 0 \qquad \text{whenever } 0 < a < b < \ell$$

then prove $f(x) \geq 0$ for every x with $0 < x < \ell$.

Problem 3 Find a solution $u = u(x,t)$ of the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u(0,t) = T_1, & t > 0 \\ u(\ell,t) = T_2, & t > 0 \end{cases}$$

where T_1 and T_2 are given constants.

Problem 4 Find as many solutions $u = u(x,t)$ as you can to the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 1 = u_x(\ell,t), & t > 0. \end{cases}$$

Do you think you have found all solutions?

Problem 5 Let $u = u(x,t)$ be a solution to the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 0 = u_x(\ell,t), & t > 0. \end{cases}$$

Show the quantity (essentially the total thermal energy)

$$\int_0^\ell u(x,t)\,dx$$

is conserved, i.e., does not change with time.

Problem 6 (Haberman 1.4.1) Find the equilibrium solution associated with the problem

$$\begin{cases} u_t = u_{xx} + x^2, & 0 < x < \ell \\ u(0,t) = T_1, & t > 0 \\ u_x(\ell,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & 0 \le x \le \ell. \end{cases}$$

Here T_1 is a given constant and u_0 is a given function.

Problem 7 (Haberman 1.4.2) Consider the equilibrium/steady state solution U of the one-dimensional heat equation on the interval $0 \le x \le \ell$ with constant conductivity K, fixed boundary temperatures $U(0) = 0 = U(\ell)$, and internal thermal energy rate-density generation/forcing modeled by $Q(x) = x$.

(a) Find an expression for the heat energy generated per unit time along the entire rod.

(b) Find an expression for the rate of heat energy flowing out of the rod at the ends $x = 0$ and at $x = \ell$.

(c) What relation should hold between your answers to the first two parts?

Problem 8 (2-D heat equation) Let U model a lamina on which the distribution of thermal energy evolves by conduction. Complete the following steps to derive the heat equation for the temperature $u : U \times [0, T) \to \mathbb{R}$:

(a) State the divergence theorem by filling in the blanks. If $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field having component functions $\mathbf{v} = (v_1, v_2)$ with continuous first partial derivatives and R is an open subset of $\mathbb{R}^2$ with closure

$$\overline{R} = R \cup \partial R \subset \underline{\hspace{2cm}}$$

and well-defined continuous outward unit normal field

$$\mathbf{n} : \partial R \to \underline{\hspace{2cm}},$$

then

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \underline{\hspace{5cm}}.$$

(b) Letting $\theta : U \times [0, T) \to \mathbb{R}$ model the **thermal energy density** in the lamina, the physical dimensions of θ are given by

$$[\theta] = \underline{\hspace{5cm}},$$

and the total thermal energy within the (sub)lamina corresponding to R is modeled by the integral expression

$$\underline{\hspace{8cm}}.$$

(c) Letting $\vec{\phi} : U \times [0, T) \to \mathbb{R}^2$ model the **thermal flux** within U, the physical dimenions of $\vec{\phi}$ are given by

$$[\vec{\phi}] = \underline{\hspace{5cm}},$$

and the integral expression

$$\int_{\partial R} \vec{\phi} \cdot \mathbf{n} \quad \text{models the rate } \underline{\hspace{6cm}} \text{ exits } \underline{\hspace{1.5cm}}.$$

(d) Assuming no independent thermal energy generation or depletion within the lamina, conservation of thermal energy is modeled by the integral equation

$$\rule{4cm}{0.4pt}$$

which by differentiating under the integral sign and using the divergence theorem may be written

$$\rule{4cm}{0.4pt} \tag{1}$$

as the vanishing of a single integral expression.

(e) Assuming

$$\frac{\partial \theta}{\partial t}$$

is continuous and $\vec{\phi}$ has component functions with continuous first spatial partial derivatives, we can use the

fundamental lemma of ________________

to conclude

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\vec{\phi}) = 0 \qquad \text{on } U \times (0, T). \tag{2}$$

Equation (2) is a _____ order partial differential equation for _____ real valued functions.

(f) The **law of specific heat** asserts $\theta = c\rho u$ where $u : U \times [0, T) \to \mathbb{R}$ models the temperature and ρ is a mass density so that

$[\rho] =$ ________________ and $[c] =$ ________________ .

(g) **Fourier's law of heat conduction** asserts

$$\vec{\phi} = K \, \rule{1.5cm}{0.4pt}$$

where K is called the conductivity and has physical units

$[K] =$ ________________ .

(h) In view of Fourier's law and the law of specific heat, the integral equation (1) may be written in terms of the gradient

$$Du = \left(\underline{\hspace{3cm}}\ ,\ \underline{\hspace{3cm}} \right)$$

as

$$\underline{\hspace{7cm}}\ ,$$

and equation (2) may be written as the ______ order partial differential equation

$$\underline{\hspace{7cm}}$$

for ______.

Problem 9 Consider the initial/boundary value problem for the 1-D heat equation:

$$\begin{cases} u_t = k u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 0, & t > 0 \\ u_x(\ell,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le \ell \end{cases} \tag{3}$$

where the initial temperature distribution is given by

$$g(x) = 1 + \cos\left(\frac{2\pi}{\ell}\, x\right).$$

(a) Solve the problem using separation of variables and Fourier series expansion.

(b) Take $\ell = 1$ and plot the solution using mathematical software as a function of two variables t and x.

(c) Take $\ell = 1$ and animate the solution using mathematical software with the time t as an animation parameter.

Problem 10 Consider the initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & (x,y,t) \in R \times (0,\infty) \\ u(x,y,0) = u_0, & (x,y) \in R \\ u(x,y,t) = 0, & (x,y,t) \in \partial R \times (0,\infty) \end{cases}$$

for the 2-D heat equation where $R = (0,4) \times (0,2)$ is a rectangular spatial domain in $\mathbb{R}^2$ and

$$u_0(x,y) = \sin\left(\frac{\pi x}{4}\right) \sin\left(\frac{\pi y}{2}\right).$$

(a) Solve the problem using separation of variables and Fourier series expansion.

(b) Animate the solution using mathematical software with the time t as an animation parameter.