Assignment 4: Heat Equation (part I)
Due Wednesday February 25, 2026

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In the following problems, let L denote a positive real number.

Problem 1 (Haberman 1.2.1-3)

(a) Determine the physical dimensions of **lineal heat energy density** $\theta = \theta(x, t)$ for which

$$\int_a^b \theta(x, t)\, dx$$

models the total heat energy in a thin rod found between positions $x = a$ and $x = b$.

(b) Consider a (more complicated) thin rod with varying cross-sectional area $A = A(x)$ and **volumetric heat energy density** also called $\theta = \theta(x, t)$. Determine the expression for the total heat energy between positions $x = a$ and $x = b$.

(c) Derive a heat equation for the temperature $u = u(x, t)$ in the rod of varying cross-sectional area from the previous part under the assumption of constant specific heat capacity c, constant volumetric density ρ, and constant heat conductivity K.

Problem 2 (Haberman 1.2.8) Give an expression for the total thermal energy in a rod modeled on an interval $0 \le x \le L$ in terms of the temperature $u = u(x, t)$.

Problem 3 Find a solution $u = u(x, t)$ of the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u(0, t) = T_1, & t > 0 \\ u(L, t) = T_2, & t > 0 \end{cases}$$

where T_1 and T_2 are given constants.

Problem 4 Find as many solutions $u = u(x,t)$ as you can to the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u_x(0,t) = 1 = u_x(L,t), & t > 0. \end{cases}$$

Do you think you have found all solutions?

Problem 5 Let $u = u(x,t)$ be a solution to the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u_x(0,t) = 0 = u_x(L,t), & t > 0. \end{cases}$$

Show the quantity (essentially the total thermal energy)

$$\int_0^L u(x,t)\,dx$$

is conserved, i.e., does not change with time.

Problem 6 (Haberman Exercise 1.3.1) One version of Newton's law of cooling states that the heat flux at the end of a thin metal rod (conducting heat) is proportional to the difference between the temperature $u(L,t)$ at the end and the **external temperature** $T = T(t)$ adjacent to the end. Use Fourier's law of heat conduction (with Newton's law of cooling) to derive an appropriate boundary flux condition for the 1-D heat equation.

Problem 7 Find the equilibrium solution associated with the problem

$$\begin{cases} u_t = u_{xx} + x^2, & 0 < x < L \\ u(0,t) = T_1, & t > 0 \\ u_x(L,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & 0 \le x \le L. \end{cases}$$

Here T_1 is a given constant and u_0 is a given function.

Problem 8 (Haberman 1.4.2) Consider the equilibrium/steady state solution U of the one-dimensional heat equation on the interval $0 \leq x \leq L$ with constant conductivity K, fixed boundary temperatures $U(0) = 0 = U(L)$, and internal thermal energy rate-density generation/forcing modeled by $Q(x) = x$.

(a) Find an expression for the heat energy generated per unit time along the entire rod.

(b) Find an expression for the rate of heat energy flowing out of the rod at the ends $x = 0$ and at $x = L$.

(c) What relation should hold between your answers to the first two parts?

Problem 9 (2-D heat equation) Let U model a lamina on which the distribution of thermal energy evolves by conduction. Complete the following steps to derive the heat equation for the temperature $u : U \times [0, T) \to \mathbb{R}$:

(a) State the divergence theorem by filling in the blanks. If $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field having component functions $\mathbf{v} = (v_1, v_2)$ with continuous first partial derivatives and R is an open subset of $\mathbb{R}^2$ with closure

$$\overline{R} = R \cup \partial R \subset \underline{\hspace{2em}}$$

and well-defined continuous outward unit normal field

$$\mathbf{n} : \partial R \to \underline{\hspace{2em}},$$

then

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \underline{\hspace{6em}}.$$

(b) Letting $\theta : U \times [0, T) \to \mathbb{R}$ model the **thermal energy density** in the lamina, the physical dimensions of θ are given by

$$[\theta] = \underline{\hspace{6em}},$$

and the total thermal energy within the (sub)lamina corresponding to R is modeled by the integral expression

$$\underline{\hspace{10em}}.$$

(c) Letting $\vec{\phi} : U \times [0, T) \to \mathbb{R}^2$ model the **thermal flux** within U, the physical dimenions of $\vec{\phi}$ are given by

$$[\vec{\phi}] = \underline{\hspace{6em}},$$

and the integral expression

$$\int_{\partial R} \vec{\phi} \cdot \mathbf{n} \quad \text{models the rate } \underline{\hspace{8em}} \text{ exits } \underline{\hspace{2em}}.$$

(d) Assuming no independent thermal energy generation or depletion within the lamina, conservation of thermal energy is modeled by the integral equation

$$\underline{\hspace{10em}}$$

which by differentiating under the integral sign and using the divergence theorem may be written

$$\underline{\hspace{10em}} \tag{1}$$

as the vanishing of a single integral expression.

(e) Assuming

$$\frac{\partial \theta}{\partial t}$$

is continuous and $\vec{\phi}$ has component functions with continuous first spatial partial derivatives, we can use the

fundamental lemma of ________________

to conclude

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\vec{\phi}) = 0 \qquad \text{on } U \times (0, T). \tag{2}$$

Equation (2) is a _____ order partial differential equation for _____ real valued functions.

(f) The **law of specific heat** asserts $\theta = c\rho u$ where $u : U \times [0, T) \to \mathbb{R}$ models the temperature and ρ is a mass density so that

$$[\rho] = \underline{\hspace{6em}} \qquad \text{and} \qquad [c] = \underline{\hspace{6em}}.$$

(g) Fourier's law of heat conduction asserts

$$\vec{\phi} = K \ \underline{\qquad}$$

where K is called the conductivity and has physical units

$$[K] = \underline{\qquad\qquad\qquad}.$$

(h) In view of Fourier's law and the law of specific heat, the integral equation (1) may be written in terms of the gradient

$$Du = \left(\underline{\qquad\qquad}, \underline{\qquad\qquad} \right)$$

as

$$\underline{\qquad\qquad\qquad\qquad\qquad},$$

and equation (2) may be written as the $\underline{\qquad}$ order partial differential equation

$$\underline{\qquad\qquad\qquad\qquad\qquad}$$

for $\underline{\qquad}$.

Problem 10 (Haberman 1.4.3) Determine the equilibrium temperature distribution $U : [0, 2] \to \mathbb{R}$ for a one-dimensional rod consisting of two different materials in perfect thermal contact at $x = 1$ and satisfying the following conditions:

(i) The material modeled on $0 \le x < 1$ has $c\rho = 1$ and $K = 1$ (where c is the specific heat capacity, ρ is the density, and K is the conductivity). Also on $0 \le x < 1$ there is an internal unit heat source with constant density per time given by $Q = 1$.

(ii) The material modeled on $1 < x \le 2$ has $c\rho = 2$ and $K = 2$ and $Q = 0$.

(iii) $U(0) = 0 = U(2)$.

Perfect thermal contact means the temperature $u = u(x)$ is continuous at $x = 1$ and the thermal energy exiting the portion of the rod modeled by $0 < x < 1$ is equal to the thermal energy entering the portion of the rod modeled by $1 < x < 2$. Be careful: This does not mean $U'(1^-) = U'(1^+)$. You need to use Fourier's law. See also Haberman 1.3.2.