Assignment 4: Linearity and Fourier series

Pace: Thursday September 26, 2024, Due Tuesday October 1, 2024

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Problem 1 (equivalent norms) Given $a, b \in \mathbb{R}$ with $a < b$, let $\| \cdot \|_{C^1} : C^1[a,b] \to \mathbb{R}$ by

$$\|f\|_{C^1} = \max_{a \le x \le b} |f(x)| + \max_{a \le x \le b} |f'(x)|$$

and $\| \cdot \|_1 : C^1[a,b] \to \mathbb{R}$ by

$$\|f\|_1 = \max \left\{ \max_{a \le x \le b} |f(x)|, \max_{a \le x \le b} |f'(x)| \right\}.$$

(a) Show $\| \cdot \|_{C^1}$ is a norm.

(b) Show $\| \cdot \|_1$ is a norm.

(c) Show $\| \cdot \|_{C^1}$ and $\| \cdot \|_1$ are equivalent norms.

Problem 2 State carefully the definition/properties of a norm, and show that given any normed linear space X with norm $\| \cdot \| : X \to [0, \infty)$ the function

$$d : X \times X \to [0, \infty) \qquad \text{by} \qquad d(x, y) = \|x - y\|$$

is a metric distance function making X a metric space.

Problem 3 State carefully the definition of a metric space and define the following terms for any metric space X:

(a) open ball Hint: Given $r > 0$ and $x \in X$, the ball $B_r(x)$ is defined by $B_r(x) =$

(b) open set Hint: A set $U \subset X$ is open if... .

(c) closed set

(d) closure of a set

(e) interior of a set

(f) convergence of a sequence

Problem 4 Let $\mathfrak{L}^1(a, b)$ denote the collection of functions $f : (a, b) \to \mathbb{R}$ for which

$$\int_{(a,b)} |f|$$

makes sense and takes a finite value. Try to show

$$\|f\|_{\mathfrak{L}^1} = \int_{(a,b)} |f|$$

defines a norm on $\mathfrak{L}^1(a, b)$.

Problem 5 Show any norm $\|\cdot\| : X \to [0, \infty)$ on a linear space X induces a metric distance function $d : X \times X \to [0, \infty)$ on X given by

$$d(x, y) = \|x - y\|.$$

You need to show

M1 $d(x, y) \geq 0$ with equality if and only if $x = y$.

M2 $d(x, y) = d(y, x)$ for all $x, y \in X$.

M3 $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

Problem 6 Let X be an **inner product space** with inner product

$$\langle \cdot \, , \, \cdot \rangle : X \times X \to \mathbb{R}$$

satisfying

IP1 $\langle x, x \rangle \geq 0$ with equality only if $x = \mathbf{0}$ is the zero vector in X.

IP2 $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$.

IP3 $\langle ax+by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ and $\langle z, ax+by \rangle = a\langle z, x \rangle + b\langle z, y \rangle$ for all $x, y, z \in X$.

You may assume the Cauchy-Schwarz inequality holds:

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \qquad \text{for all } x, y \in X.$$

(a) Show $\langle x, y \rangle \leq \|x\| \, \|y\|$ for all $x, y \in X$.

(b) Show that equality holds in part **(a)** if and only if $\{x, y\}$ is a linearly dependent set in X.

Problem 7 Use the trigonometric identity

$$\cos\left(\frac{j\pi}{L}\, x\right) \cos\left(\frac{k\pi}{L}\, x\right) = \frac{1}{2}\left[\cos\left(\frac{(j+k)\pi}{L}\, x\right) + \cos\left(\frac{(j-k)\pi}{L}\, x\right)\right]$$

to evaluate

$$\int_{-L}^{L} \cos\left(\frac{j\pi}{L}\, x\right) \cos\left(\frac{k\pi}{L}\, x\right) \, dx$$

for $j, k = 0, 1, 2, 3, \ldots$.

Similarly, use

$$\sin\left(\frac{j\pi}{L}\, x\right) \cos\left(\frac{k\pi}{L}\, x\right) = \frac{1}{2}\left[\sin\left(\frac{(j+k)\pi}{L}\, x\right) + \sin\left(\frac{(j-k)\pi}{L}\, x\right)\right]$$

to evaluate

$$\int_{-L}^{L} \sin\left(\frac{j\pi}{L}\, x\right) \cos\left(\frac{k\pi}{L}\, x\right) \, dx$$

for $j, k = 0, 1, 2, 3, \ldots$.

What would you do with

$$\int_{-L}^{L} \sin\left(\frac{j\pi}{L}\, x\right) \sin\left(\frac{k\pi}{L}\, x\right) \, dx \qquad ?$$

Problem 8 (even periodic Fourier series) Find the Fourier expansion of the function $f \in C^0[-L, L]$ given by

$$f(x) = L^2 - x^2.$$

(a) Plot the function.

(b) Consider the formal expansion

$$f(x) = a_0 + \sum_{j=1}^{\infty} a_j v_j + \sum_{j=1}^{\infty} b_j w_j \tag{1}$$

with

$$v_j(x) = \cos\left(\frac{j\pi}{L}\, x\right) \qquad \text{and} \qquad w_j(x) = \sin\left(\frac{j\pi}{L}\, x\right).$$

Integrate $f v_j$ and $f w_j$ and use the orthogonality properties of the basis functions v_j and w_j in $\mathfrak{L}^2(-L, L)$ to find the coefficients.

(c) Choose some specific value for L and plot the first few terms of the resulting series (using mathematical software) to confirm you have found the correct formula for the coefficients.

Problem 9 (odd periodic Fourier series) Find the Fourier expansion of the function $f \in C^0[-\pi, \pi]$ given by

$$f(x) = \begin{cases} \left|x + \dfrac{\pi}{2}\right| - \dfrac{\pi}{2}, & -\pi \le x \le 0 \\[2ex] \dfrac{\pi}{2} - \left|x - \dfrac{\pi}{2}\right|, & 0 \le x \le \pi \end{cases}$$

(a) Plot the function.

(b) For $L = \pi$ consider the formal expansion (1) of Problem 8. Integrate $f v_j$ and $f w_j$ and use the orthogonality properties of the basis functions v_j and w_j in $\mathfrak{L}^2(-\pi, \pi)$ to find the coefficients.

(c) Plot the first few terms of the resulting series (using mathematical software) to confirm you have found the correct formula for the coefficients.

Problem 10 (periodic Fourier series) Find the Fourier expansion of the function $f \in C^0[-1,1]$ given by

$$f(x) = \begin{cases} x^2, & -1 \leq x \leq 0 \\ x, & 0 \leq x \leq 1 \end{cases}$$

(a) Plot the function.

(b) For $L = 1$ consider the formal expansion (1) of Problem 8. Integrate fv_j and fw_j and use the orthogonality properties of the basis functions v_j and w_j in $\mathfrak{L}^2(-\pi,\pi)$ to find the coefficients.

(c) Plot the first few terms of the resulting series (using mathematical software) to confirm you have found the correct formula for the coefficients.