Assignment 3: Fourier Series
Due Wednesday February 11, 2026

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In the following problems, let L denote a positive real number.

Problem 1 Find the Fourier basis for $L^2(0, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = 0 = \phi(L). \end{cases} \tag{1}$$

Problem 2 Find the Fourier basis for $L^2(0, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0 = \phi'(L). \end{cases} \tag{2}$$

Problem 3 Find the Fourier basis for $L^2(0, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi'(0) = 0 = \phi(L). \end{cases} \tag{3}$$

Problem 4 Find the Fourier basis for $L^2(-L, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(-L) = 0 = \phi(L). \end{cases} \tag{4}$$

Problem 5 Find the Fourier basis for $L^2(0, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(L) \\ \phi'(0) = \phi'(L). \end{cases} \tag{5}$$

This is a case where it can be helfpful for organization to do some renaming of the eigenfunctions.

Problem 6 (Haberman 3.2.2 part (a)) Complete the following steps to expand the function with values $f(x) = x$ in terms of the Fourier basis for $L^2(-L, L)$ associated with the Sturm-Liouville problem

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L). \end{cases} \tag{6}$$

I will outline the steps using the notation for a regular Sturm-Liouville problem with $\phi_1, \phi_2, \phi_3, \ldots$ for the eigenfunction basis with ϕ_j corresponding to the eigenvalue λ_j for $j = 1, 2, 3, \ldots$ and $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Caution: Since the problem is not regular and you may have done some renaming for the sake of organization, you may have to adapt the ideas though the notation may not apply directly.

(a) Assume

$$f(x) = \sum_{j=1}^{\infty} a_j \phi_j.$$

Multiply by a fixed basis function, integrate both sides from $-L$ to L, and use orthogonality to solve for the coefficients.

(b) Pick a specific positive value for L and use mathematical software to plot partial sums on $[-L, L]$ to check/verify you have found the correct coefficients.

(c) Use mathematical software to plot partial sums on $[-2L, 2L]$ to see what you get and conjecture values for the limits

$$\lim_{k\to\infty} \sum_{j=1}^{k} a_j \phi_j(-L) \qquad \text{and} \qquad \lim_{k\to\infty} \sum_{j=1}^{k} a_j \phi_j(L).$$

Prove your conjecture.

Problem 7 Repeat Problem 6 using the Fourier basis for $L^2(0, L)$ from Problem 5. What differences do you notice? (Though the values of the function f which is being expressed as Fourier series are only used on a specific interval for example $(0, L)$ or $(-L, L)$, the Fourier series expansions define functions on all of $\mathbb{R}$.)

Problem 8 Repeat Problem 6 using the Fourier basis for $L^2(-L, L)$ from Problem 4. What differences do you notice? (Though the function f is continuous and even very smooth on $[-L, L]$, the function defined by a Fourier expansion may not be continuous.)

Problem 9 (Haberman 3.2.1 parts (a) and (b)) In each part of this problem check your work by use mathematical software to plot partial sums for a specific value of L.

(a) Expand the constant function with values $f(x) = L^2$ using the Fourier basis for $L^2(0, L)$ from Problem 1.

(b) Expand the linear function with values $g(x) = Lx$ using the Fourier basis for $L^2(0, L)$ from Problem 2.

(c) Expand the function with values $h(x) = x(L - x)$ using the Fourier basis for $L^2(0, L)$ from Problem 2. Hint: Find the expansion for the function with values $f(x) = x^2$ and then use linearity and part **(b)** noting $h = g - f$.

Problem 10 (Haberman 3.3.2 part (d)) In each part of this problem check your work by use mathematical software to plot partial sums for a specific value of L. At some point you may wish to use the symbolic integration facilities of your mathematical software to either check your integrations or simply to do the integration for you.

(a) Find the Fourier series expansion of the function with values

$$f(x) = \begin{cases} 0, & x < L/2 \\ 1, & x > L/2. \end{cases}$$

Use the expansion basis from Problem 1.

(b) Repeat part **(a)** using the basis from Problem 3.

(c) Find the Fourier series expansion of the function with values

$$g(x) = \begin{cases} 1, & x < L/2 \\ 0, & x > L/2. \end{cases}$$

Use the expansion basis from Problem 3.