Assignment 3: Spaces of functions; linearity

Pace: Thursday September 19, 2024, Due Tuesday September 24, 2024

John McCuan

The following definitions are used in the problems below. You may wish to read them carefully before you try the problems.

Definition 1 (open set in $\mathbb{R}^n$) A set $U \subset \mathbb{R}^n$ is **open** if for each $\mathbf{p} \in U$ there exists some $\delta > 0$ such that

$$B_\delta(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\} \subset U.$$

A set of the form $B_\delta(\mathbf{p})$ is called a **ball** with center $\mathbf{p}$ and radius δ. The boundary of $B_\delta(\mathbf{p})$ is

$$\partial B_\delta(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = \delta\}.$$

Definition 2 (closed set in $\mathbb{R}^n$) A set $A \subset \mathbb{R}^n$ is **closed** if the *complement*

$$A^c = \mathbb{R}^n \backslash A = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin A\}$$

is open.

Problem 1 (regularity) Find a function in $C^1[-1,1]\backslash C^2[-1,1]$. This means the function should be in $C^1[-1,1]$ but not in $C^2[-1,1]$, so that $C^2[-1,1]$ is a **proper subspace** of $C^1[-1,1]$.

Problem 2 (regularity: real analyticity versus smooth) Find a function in

$$C^\infty(\mathbb{R})\backslash C^\omega(\mathbb{R}).$$

Problem 3 (open sets, closed sets, and closure) Consider the following seven sets

(i) $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$.

(ii) $[0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$.

(iii) $(-\infty, 0) = \{x \in \mathbb{R} : x < 0\}$.

(iv) $(-\infty, 0] = \{x \in \mathbb{R} : x \le 1\}$.

(v) $\mathbb{R}$

(vi) $\mathbb{R}^n$

(vii) ϕ (The **empty set** is the unique set with no elements. The empty set ϕ is a subset of every set.)

(a) Determine if each of the seven sets above is open.

(b) Determine if each of the seven sets above is closed.

(c) Show *any* intersection of closed sets is closed. This means that if $\{A_\alpha\}_{\alpha\in\Gamma}$ is a collection of closed sets indexed by any nonempty set Γ, then the intersection

$$A = \bigcap_{\alpha\in\Gamma} A_\alpha$$

is closed. Note: The indexing set does not have to be finite here or even countable, but it does have to be nonempty. For example, one could consider

$$\bigcap_{r>0} B_r(\mathbf{0}).$$

(d) Use part **(c)** to show the following:

> *Given any set A there exists a unique smallest closed set containing A.*

Hint: Express the set asserted to exist in terms of an intersection of closed sets. Be careful however: Empty intersections are not allowed.

The smallest closed set containing a set A is called the **closure** of A and is denoted by $\overline{A}$.

Problem 4 (support) The **support** of a real valued function $u : A \to \mathbb{R}$ is

$$\text{supp}(u) = \overline{\{x \in A : u(x) \neq 0\}}.$$

The closure on the right is discussed in Problem 3 above. Consider the situation in which the following conditions hold:

(i) The domain of u is an open subset U of $\mathbb{R}^n$, so that $u : U \to \mathbb{R}$,

(ii) $\text{supp}(u) \subset U$, and

(iii) $\text{supp}(U)$ is *bounded* in the sense that for some $R > 0$ there holds

$$\text{supp}(u) \subset B_R(\mathbf{0}).$$

When these conditions hold, we say the function $u : U \to \mathbb{R}$ is **compactly supported** in U and write $\text{supp}(u) \subset\subset U$. (A closed and bounded subset of $\mathbb{R}^n$ is said to be *compact.*)

Show the function $\phi_0 : \mathbb{R}^n \to [0, 1]$ given by

$$\phi_0(\mathbf{x}) = \begin{cases} e^{(|\mathbf{x}|^2 - 1)^{-1}}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \geq 1 \end{cases}$$

satisfies **(a)** $\phi_0 \in C^\infty(\mathbb{R}^n)$, **(b)** ϕ_0 is compactly supported in $\mathbb{R}^n$, and **(c)**

$$\text{supp}(\phi_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}. \tag{1}$$

Problem 5 (open and closed balls; boundary)

(a) Show $B_\delta(\mathbf{p})$ is open. For this reason the ball $B_\delta(\mathbf{0})$ is often called an *open* ball.

(b) The **boundary** of a set $A \subset \mathbb{R}^n$ is

$$\partial A = \overline{A} \cap \overline{A^c}.$$

Express $\partial B_r(\mathbf{p})$ in "set builder" notation like the set on the right in (1).

The set $\overline{B_r(\mathbf{p})}$ is called the *closed ball* of radius r and center $\mathbf{p}$. The set $\partial B_r(\mathbf{p})$ is called the *sphere* of radius r and center $\mathbf{p}$.

(c) Express each of the seven sets in Problem 3 in terms of one or more balls.

Problem 6 (test functions) Given an open set $U \subset \mathbb{R}^n$, the collection of all functions $\phi \in C^\infty(U)$ with

$$\operatorname{supp}(\phi) \subset\subset U \tag{2}$$

is denoted $C_c^\infty(U)$. Assuming U is a nonempty set, show $C_c^\infty(U) \neq \phi$. In particular given $\mathbf{p} \in U$ and $r > 0$ with $\overline{B_r(\mathbf{p})} \subset U$, construct a function $\phi_1 \in C_c^\infty(U)$ satisfying the following

(a) $\phi_1 \geq 0$,

(b) $\operatorname{supp}(\phi_1) = \overline{B_r(\mathbf{p})}$, and

(c)

$$\int \phi = 1.$$

Hint: Start with the case $n = 1$ and $\mathbf{p} = 0 \in U = \mathbb{R}$.

In the special case $\mathbf{p} = \mathbf{0} \in U = \mathbb{R}^n$, the function ϕ_1 one constructs to complete this problem is called a **standard test function**. A standard test function is also usually assumed to satisfy $\phi_1(-\mathbf{x}) = \phi_1(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$; you can check to see if the function you construct satisfies this condition in the special case. More generally $C_c^\infty(U)$ is called a space of **test functions**.

The line (2) is read: "The support of ϕ is **compactly contained** in (the open set) U.

Problem 7 (mollification) There is an interesting operation called *mollification* which associates to a potentially "not very regular" function $u : \mathbb{R}^n \to \mathbb{R}$ a family of very regular functions $\{u * \mu_\delta\}_{\delta>0} \subset C^\infty(\mathbb{R}^n)$.

Given a standard test function $\phi_1 \in C_c^\infty(\mathbb{R}^n)$, the family of funcitons

$$\{\phi_\delta\}_{\delta>0} \subset C_c^\infty(\mathbb{R}^n)$$

with

$$\phi_\delta(\mathbf{x}) = \frac{1}{\delta^n}\phi_1\left(\frac{\mathbf{x}}{\delta}\right)$$

is called a **family of approximate identity**. For each $\delta > 0$ we define the **mollification** of u to be the function $u * \phi_\delta : \mathbb{R}^n \to \mathbb{R}$ by

$$u * \phi_\delta(\mathbf{x}) = \int_{\xi\in\mathbb{R}^n} u(\xi)\ \phi_\delta(\mathbf{x} - \xi).$$

The idea is that in some sense

$$\lim_{\delta\searrow 0} u * \phi_\delta = u,$$

but we won't be able to make proper sense of this limit until a little later. For now we consider some simpler aspects of the mollification process.

(a) Show

$$\int_{\mathbb{R}^n} \phi_\delta = 1 \qquad \text{for all } \delta > 0.$$

(b) Calculate explicitly $h * \phi_\delta(x)$ for $|x| > \delta$ where $h : \mathbb{R} \to \mathbb{R}$ is the **Heaviside function** given by

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

(c) Use mathematical software to plot the mollification of the Heaviside function.

Problem 8 (power series; radius of convergence) Find the Taylor series expansion

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}\; x^j \tag{3}$$

for the function $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1-x}.$$

(a) Is it true that

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}\; x^j \tag{4}$$

for all $x \in \mathbb{R}\backslash\{1\}$? If not, find the convergence set

$$C = \left\{ x \in \mathbb{R}\backslash\{1\} : f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}\; x^j \right\}.$$

(b) What can you say about the value of the series (3) when $x \in \mathbb{R}\backslash C$?

(c) Given an open set $U \subset \mathbb{R}^n$, a real analytic function $u \in C^\omega(U)$, and a center of expansion $\mathbf{p} \in U$, the largest value $r > 0$ for which

$$u(\mathbf{x}) = \sum_{|\beta|=0}^{\infty} \frac{D^\beta(\mathbf{p})}{\beta!}\; (\mathbf{x} - \mathbf{p})^\beta \tag{5}$$

holds for all $\mathbf{x}x \in B_r(\mathbf{p})$ is called the **radius of convergence** at $\mathbf{p}$. Find the radius of convergence of the series you found for the function $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ from part **(a)** at each $x_0 \in \mathbb{R}\backslash\{1\}$.

Problem 9 (multivariable power series) Find the power series expansion of $u : \mathbb{R}^3 \to \mathbb{R}$ by

$$u(x, y, z) = \sin x \sin y \sin z$$

at $\mathbf{p} = (0, 0, 0)$. Hint(s): The formulas

$$\frac{d^{4k}}{dx^{4k}} \sin x = \sin x$$
$$\frac{d^{4k+1}}{dx^{4k+1}} \sin x = \cos x$$
$$\frac{d^{4k+2}}{dx^{4k+2}} \sin x = -\sin x$$
$$\frac{d^{4k+3}}{dx^{4k+3}} \sin x = -\cos x$$

for $k = 0, 1, 2, 3, \ldots$ give all the derivatives of $\sin x$. The answer is

$$\sum_{|\beta|=0}^{\infty} \frac{1}{\beta!} \frac{(-1)^{\frac{|\beta|-3}{2}}}{8} \; [1 - (-1)^{\beta_1}][1 - (-1)^{\beta_2}][1 - (-1)^{\beta_3}] \; x^{\beta_1} y^{\beta_2} z^{\beta_3}$$

$$= xyz - \frac{1}{6}(x^3yz + xy^3z + xyz^3) + \frac{1}{36}(x^3y^3z + x^3yz^3 + xy^3z^3) - \frac{1}{216}x^3y^3z^3 + \cdots$$

what is

$$\frac{d^{\beta_1}}{dx^{\beta_1}} \sin x\Big|_{x=0}?$$

Problem 10 How many multiindices $\beta \in \mathbb{N}_0^n$ are there with $|\beta| = k$ and how many partial derivatives are there of a function $u \in C^k(U)$? Are these the same number? Hint: Start with $n = 1$ and increase dimension from there.