Assignment 2: ODE
Pace: Thursday September 5, 2024, Due Tuesday September 10, 2024

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Problem 1 below concerns the following result:

Theorem 1 (existence and uniqueness theorem for a BVP) Given $L > 0$ and $g \in C^0[0, L]$, the problem

$$\begin{cases} u'' = g(x), \quad x \in (0, L) \\ u(0) = 0, \\ u(L) = 0. \end{cases} \tag{1}$$

has a unique solution $u \in C^1[0, L]$.

Problem 1 (two point BVP) Prove Theorem 1.

Problem 2 (power series solution) Find the general solution of the ODE $y'' + x^2 y = 0$ as a power series. Hint(s): Your solution should contain two arbitrary constants c_1 and c_2, and you should be able to write it in the form $y = c_1 y_1 + c_2 y_2$ where $\{y_1, y_2\} \subset C^\omega(\mathbb{R})$ is a **basis of solutions**.

Problem 3 (linear ODE) An n-th order **linear ordinary differential operator** is a function $L : C^\omega(a, b) \to C^\omega(a, b)$ of the form

$$Ly = \sum_{j=0}^{n} a_j(x) y^{(j)}$$

for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and some coefficients $a_j \in C^\omega(a, b)$ with $a_n(x) \equiv 1$. Show that such an operator satisfies

(a) $L[y_1 + y_2] = Ly_1 + Ly_2$ for every $y_1, y_2 \in C^\omega(a, b)$, and

(b) $L[cy] = cLy$ for every $y \in C^\omega(a, b)$ and $c \in \mathbb{R}$.

Problem 4 (linear ODE: IVP) A **linear ordinary differential equation of order** n is an equation of the form

$$Ly = f \tag{2}$$

where $f \in C^\omega(a, b)$ and L is an n-th order linear operator on $C^\omega(a, b)$; see Problem 3 above.

(a) Find a first order system $\mathbf{x}' = \mathbf{F}(\mathbf{x}, t)$ of ordnary differential equations of the type considered in the general existence and uniqueness theorem for ODEs and the existence theorem for linear ODE which is equivalent to the single n-th order ODE (2).

(b) What does the existence theorem for linear ODE tell you about the equation (2)?

- **(i)** What do you know about existence?
- **(ii)** What do you know about uniqueness?
- **(iii)** What are the appropriate initial conditions to consider along with equation (2)?

(c) If y_1 and y_2 are solutions of (2) is $y_1 + y_2$ a solution?

Problem 5 (linear ODE) A linear ordinary differential equation $Ly = f$ is said to be **homogeneous** if $f \equiv 0$; see Problem 4 above. Let $L : C^\omega(a, b) \to C^\omega(a, b)$ be a linear ordinary differential operator of order n as considered in Problems 3 and 4 above.

(a) Show that if y_1 and y_2 are solutions of a homogeneous linear ODE $Ly = 0$, then any linear combination of y_1 and y_2 is also a solution.

(b) Show the solution set

$$\Sigma_0 = \{y \in C^\omega(a, b) : Ly = 0\}$$

is a vector space and find the dimension of this vector space.

(c) In the equation $Ly = f$, especially when f is not the zero function, the function $f \in C^\omega(a, b)$ is called the **inhomogeneity**. Show that if $y_p \in C^\omega(a, b)$ satisfies $Ly_p = f$, then every other element y of the solution set

$$\Sigma = \{y \in C^\omega(a, b) : Ly = f\}$$

satisfies $y = y_p + y_h$ for some $y_h \in \Sigma_0$ from part **(b)**. Hint: Consider $y - y_p$ and show $y - y_p \in \Sigma_0$.

Given the equation $Ly = f$, the equation $Ly = 0$ is called the **associated homogeneous equation**.

Problem 6 (Sturm-Liouville type problems) Given $L > 0$, solve the two point boundary value problem

$$\begin{cases} y'' = -y, \quad x \in (0, L) \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$

Hint: Consider various cases based on the value of L.

Problem 7 (Fourier series solution) Solve the BVP

$$\begin{cases} y'' = \sin(\pi x) + 2\sin(2\pi x) + 3\sin(3\pi x), \quad x \in (0, 1) \\ y(0) = 0 \\ y(1) = 0. \end{cases} \tag{3}$$

Problem 8 (Problem 7) Is your solution in Problem 7 unique or did you find multiple solutions?

Problem 9 (Problem 7) Complete the following steps to "show" (or at least suggest) the inhomogeneity in Problem 7 has a unique Fourier sine series representation:

(a) Assume

$$\sin(\pi x) + 2\sin(2\pi x) + 3\sin(3\pi x) = \sum_{j=1}^{\infty} b_j \sin(j\pi x).$$

Multiply both sides of this equation by $\sin(\pi x)$ and integrate from $x = 0$ to $x = 1$. Solve for b_1.

(b) Similarly, solve for b_2 and b_3.

(c) Show explicitly that $b_j = 0$ for $j = 4, 5, 6, \ldots$.

Problem 10 (Sturm-Liouville problem) Given $L > 0$ consider the **one parameter family** of two point boundary value problems

$$\begin{cases} y'' + \lambda y = 0, \quad x \in (0, L) \\ y(0) = 0, \\ y(L) = 0. \end{cases} \tag{4}$$

Consider three cases in order to find all "nontrivial," i.e., nonzero, solutions.

(a) $\lambda < 0$.

(b) $\lambda = 0$.

(c) $\lambda > 0$.