Assignment 1: ODE
Partial Solutions and Comments

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In the problems below we refer to the following results:

Theorem 1 (general local existence and uniqueness) If

$$\mathbf{F} \in C^1(\mathbb{R}^n \times (a, b) \to \mathbb{R}^n),$$

then for any $\mathbf{p} \in \mathbb{R}^n$ and any $t_0 \in (a, b)$ there exists some $\epsilon > 0$ such that the initial value problem (IVP)

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}, t) & t_0 - \epsilon < t < t_0 + \epsilon \\ \mathbf{x}(t_0) = \mathbf{p} \end{cases} \tag{1}$$

has a unique solution.

Theorem 2 (existence and uniqueness theorem for linear ODE) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. If $a_{ij}, b_j \in C^0(a, b)$ for $i, j = 1, 2, \ldots, n$, then for every $(\mathbf{p}, t_0) \in \mathbb{R}^n \times (a, b)$ the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{b}, & t \in (a, b) \\ \mathbf{x}(t_0) = \mathbf{p}, \end{cases} \tag{2}$$

where $A \in C^0((a, b) \to \mathbb{R}^{n \times n})$ is the $n \times n$ matrix malued function with the real valued function a_{ij} in the i-th row and j-th column and $\mathbf{b} \in C^0((a, b) \to \mathbb{R}^n)$ is the vector valued function with j-th component function b_j, has a unique solution $\mathbf{x} \in C^1((a, b) \to \mathbb{R}^n)$.

Problem 1 (continuity) Let a and b be extended real numbers in $\mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and let U be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N} = \{1, 2, \ldots\}$ (the natural numbers).

(a) State carefully the definition of continuity for a function $f : (a, b) \to \mathbb{R}$.

(b) If $f, g \in C^0(U)$, show $f + g \in C^0(U)$.

(c) If $f \in C^0(U)$, show $cf \in C^0(U)$ for every $c \in \mathbb{R}$.

These are the two main properties making $C^0(U)$ a **vector space**.

Problem 2 (initial value problem) If $\mathbf{x}$ is the solution of the initial value problem in the general existence and uniqueness theorem for ODEs, then it is natural to assume $\mathbf{x}$ is **differentiable**. Show that in fact, under the assumptions of the theorem the solution is **continuously differentiable** that is $\mathbf{x} \in C^1((t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n)$.

Problem 3 (an ODE) Solve the IVP:

$$\begin{cases} \mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} \\ \\ \mathbf{x}(t_0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{cases}$$

Problem 4 (IVP) Consider the initial value problem

$$\begin{cases} y'' = y^2 \\ y(0) = 3. \end{cases} \tag{3}$$

(a) What does Theorem 1 tell you about the solutions to this problem?

(b) Explore the following assertion numerically:

> There exists a uniform $\epsilon > 0$ for which all solutions of (3) are well-defined and unique on the interval $(-\epsilon, \epsilon)$, that is for $-\epsilon < x < \epsilon$.

(c) Multiply both sides of the ODE by y' and integrate to obtain an implicit solution. Does this tell you anything decisive about the assertion of part **(b)**?

Problem 4 Solution:

(a) What does Theorem 1 tell you about the solutions to this problem?

If we replace the second order equation with a first order system for a vector valued function $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))^T$:

$$\begin{cases} \mathbf{x}' = (\mathbf{x}) = \begin{pmatrix} x_2 \\ x_1^2 \end{pmatrix}, \quad x \in \mathbb{R} \\ \\ x_1(0) = 3 \\ x_2(0) = v, \end{cases}$$

then we may observe first that the original problem does not provide a specification for the value of $y'(0) = v$. Nevertheless, taking any $v \in \mathbb{R}$, the theorem tells us the following:

(i) For any $v \in \mathbb{R}$ there exists some $\delta > 0$ for which the system has a unique solution $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{x} \in C^1((-\delta, \delta) \to \mathbb{R}^2)$ and, consequently, the original initial value problem for the second order equation has a unique solution $y = y(x)$ with $y \in C^2(-\delta, \delta)$.

Note 1 Notice that for the equivalent system we have changed the name of the independent variable so that $-\delta < t < \delta$, while in the original problem we obtain a solution defined for $-\delta < x < \delta$. This change of variable name may seem unnecessary in this case because the equation is autonomous. I guess that in fact Theorem 1 allows nonautonomous structural functions $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$, so one can definitely use x as the independent variable in place of t in the statement of the theorem. If one does that, however, one ends up with things like $\mathbf{x}(x)$ and $\mathbf{F}(\mathbf{x}, x)$ which may be a little notationally confusing. Furthermore, the special case of Theorem 1 for autonomous structure functions $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is essentially equivalent to the statement as we have it. (You can think about why that is the case.) And if you wanted to apply the autonomous version of the theorem to a nonautonomous equation like $y'' = xy^2$, then you would probably want to introduce x as one of the independent variables: $x_1 = x$, $x_2 = y$, $x_3 = y'$. Then it may be convenient to maintain the distinction between the independent variables x for the original equation/initial value problem and t for the equivalent system.

Note 2 One of the main points of this problem is to bring to your attention that the number δ depends on the value v introduced as a value of $y'(0)$. Theorem 1 also tells you it is natural to introduce this value.

(ii) In light of **(i)** the theorem tells us the original problem does have some solution y on some interval $-\delta < x < \delta$. The solution y clearly depends on the value $y'(0)$, and so the original problem clearly has many different solutions corresponding to all the different possible values of $v = y'(0)$, and each of these (as far as we know) exists on a different interval $-\delta(v) < x < \delta(v)$ or more generally on an interval $a(v) < x < b(b)$ with $a(v) \le -\delta(v) < \delta(v) \le b(v)$. While the solution for a given value $v = y'(0)$ is unique, the solution/solutions of the original problem is/are not unique.

(b) Explore the following assertion numerically:

> There exists a uniform $\epsilon > 0$ for which all solutions of (3) are well-defined and unique on the interval $(-\epsilon, \epsilon)$, that is for $-\epsilon < x < \epsilon$.

Perhpas the most natural starting point for this problem is careful consideration of the nonlinear first order equation/system $y' = y^2$, $y(x_0) = y_0$ considered in the in-class/recorded Lecture 3 and in Problem 8 below.[1] For the equation $y' = y^2$ with $y(0) = y_0$ if we assume for simplicity that $y_0 > 0$, then the solution guaranteed by Theorem 1 will be increasing for $x \ge 0$ since $y^2 \ge 0$ (always) meaning y increases from a positive value y_0 and will remain positive for all $x \ge 0$. In fact, it will be the case that $y'(x) > 0$ for all x for a somewhat different reason: If there were some $x_* < 0$ for which $y'(x_*) = 0$, then we would have also $y(x_*) = 0$ as well. We can then naturally take

$$x_* = \max\{x : y(x) = 0\}. \tag{4}$$

Applying Theorem 1 to the problem

$$\begin{cases} y' = y^2 \\ y(x_*) = 0 \end{cases}$$

we obtain a contradiction because $y \equiv 0$ is the unique solution, and we know there is some $\delta > 0$ for which $y(x) = 0$ for $0 \le x < \delta$. This contradicts the fact

[1] By the end of this solution, I will have also discussed this problem enough to constitute at least most aspects of a solution of Problem 8 below, and you may want to go and try Problem 8 on your own before reading further.

(4) that x_* is the maximum. Thus, there is no x_* with $y'(x_*) = 0$, and $y'(x) > 0$ for all x. This means $y(x) > 0$ for all x as well, and we have no problem writing the equation as

$$\frac{y'}{y^2} = 1$$

on whatever interval $a < x < b$ where the solution is defined. Thus, we can find a formula for the solution:

$$-\frac{1}{y} + \frac{1}{y_0} = x \qquad \text{or} \qquad y = -\frac{1}{x - 1/y_0}. \tag{5}$$

From this formula, we see exactly what happens. We have $a = -\infty$ and $b = 1/y_0$, so that the solution tends to $+\infty$ at the right endpoint. More explicitly,

$$\lim_{x \nearrow 1/y_0} y(x) = +\infty.$$

In particular, if we restrict back to a symmetric interval of existence $-\delta < x < \delta$, then the largest $\delta = \delta(y_0)$ we can take is $\delta = 1/y_0$. Clearly, there does not exist an $\epsilon > 0$ such that all solutions of this problem are defined on the interval $-\epsilon < x < \epsilon$. This is because such a value ϵ would have to satisfy $\epsilon \leq \delta(y_0)$ for all y_0, and

$$\epsilon \leq \lim_{y_0 \nearrow \infty} \frac{1}{y_0} = 0. \tag{6}$$

The conclusion from this discussion might be something like this:

> If I have an ODE with the derivative y' prescribed as the square y^2 of the value y, that is if the function grows at a rate given by the square of the value of y, then the function blows up in finite time, i.e., for a finite value of x namely $1/y_0$ when $y_0 > 0$, and that finite time of blow-up $\delta = b = 1/y_0$ can be arbitrarily small if the initial value y_0 becomes (arbitrarily) large, which it can.

Figure 1 shows the solutions corresponding to $y_0 = 1/2$ with blow-up at $x = 2$, $y_0 = 1$ with blow-up at $x = 1$, and $y_0 = 2$ with blow-up at $x = 1/2$. In fact, the "time" of blow-up $x = \delta(y_0)$ is monotonically decreasing in y_0 for $y_0 > 0$.

Note: As mentioned above, if $y_0 = 0$, then the unique solution is $y(x) \equiv 0$ which is well-defined for all $x \in \mathbb{R}$. If $y_0 < 0$, then the analysis is somewhat different with a blow-up at the left endpoint $x = a = -\delta$ of the maximal interval (a, ∞) of definition of the solution.

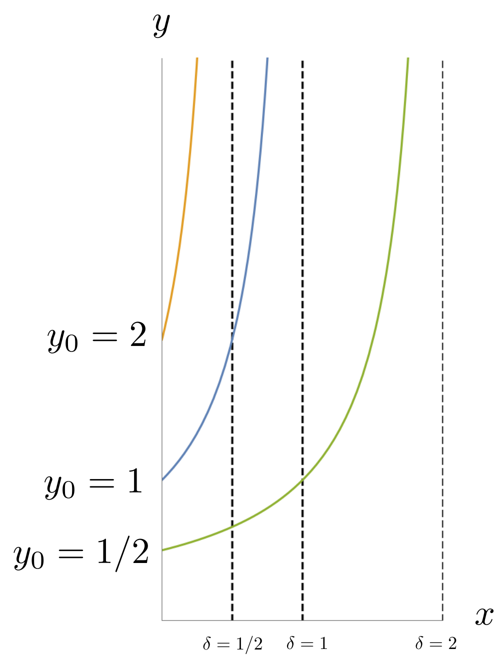

Figure 1: Solutions of $y' = y^2$, $y(0) = y_0$ illustrating finite time blow-up and the monotone dependece of the blow-up time on the initial value.

In this problem we have $y'' = y^2$ where the first derivative y' grows like the value y^2 because $y'' = (y')'$. This is somewhat more complicated, but assuming the value $v = y'(0) > 0$, the value of $y(x)$ should always be at least 3 for $x \geq 0$, and with the second derivative given by the square of the value (and the first derivative growing at a rate given by the square of the value) it is not unreasonable to **guess** the solution will display a finite time blow-up. In fact, we might observe at this point that our answer to part **(a)** can be expanded as follows:

For $v \geq 0$, the solution y (as well as the derivative y') will be increasing for $x \geq 0$ and strictly increasing for $x > 0$. **Increasing functions always have limits either finite or infinite.** In this case, if we assume a maximum interval (a, b) of existance for the problem with $0 < b < \infty$ and we also assume

$$3 < L_0 = \lim_{x \nearrow b} y(x) < \infty,$$

then we must also have

$$L_2 = \lim_{x \nearrow b} y''(x) = \lim_{x \nearrow b} y(x)^2 = L_0^2,$$

and consequently, we must have

$$L_1 = \lim_{x \nearrow b} y'(x) < \infty$$

as well. This means we can consider the initial value problem

$$\begin{cases} y'' = y^2, & x \in \mathbb{R} \\ y(b) = L_0, \\ y'(b) = L_1 \end{cases}$$

or the equivalent system

$$\begin{cases} \mathbf{x}' = (\mathbf{x}) = \begin{pmatrix} x_2 \\ x_1^2 \end{pmatrix}, & x \in \mathbb{R} \\ \\ x_1(b) = L_0 \\ x_2(b) = L_1, \end{cases}$$

and obtain an extension of the solution from Theorem 1 contradicting the assumption that the maximal interval of existence (a, b) extended only to a finite value b on the right. Thus, either there will be a finite value $b = \delta(v) > 0$ for which the solution is defined for $0 \le x < b$ with

$$\lim_{x \nearrow \delta} y(x) = \infty,$$

i.e., there will be finite time blow-up on the right, **or** the solution will be defined on an interval (a, ∞). In summary, the only possible reason for a solution of the equation $y'' = y^2$ with $y(0) = 3$ to have an interval of existence bounded on the right is that the solution displays finite time blow-up (something) like we saw with the equation $y' = y^2$.

One might also **guess** at this point that not only do the solutions for $v \ge 0$ blow up in finite time, but also that the time of blow up $b = \delta(v)$ is a monotone decreasing function of v with

$$\lim_{v \nearrow \infty} \delta(v) = 0 \tag{7}$$

much like we saw in (6). In particular, if this is the case, it would preclude the existence of a value $\epsilon > 0$ as described in the main assertion of part **(b)**. Thus, our guess might be that the assertion is not correct.

Let's see if we can see any numerical evidence for these guesses. I can find a numerical approximation for the solution of the initial value problem (with the additional condition $y'(0) =$ `v` on the interval from $x = 0$ to `delta`, if it has one, with the Mathematica function

```
soln[v_, delta_] := NDSolve[ { odey'[odex] == odev[odex],
          odev'[odex] == odey[odex]^2, odey[0] == 3, odev[0]== v},
                    {odey,odev},{odex,0,delta}]
```

Note: "soln" is the name of the function. The ":=" is used so that the command is not evaluated until specific numerical values for `v` and `delta` are given. This can be further augmented by the use of `soln[v_?NumericQ,delta_?NumericQ]`. The "double equals" `==` are important to use for the input of the equations and the initial values. I use the alternative symbol `odex` for the independent variable because sometimes Mathematica gets confused if there is an independent variable called `x` inside the ODE solver `NDSolve` and another one used outside. I used the alternative symbols `odey` and `odev` for y and y' respectively for reasons which hopefully will become clear momentarily.

In order to plot solutions and find specific values, I define the function

```
y[x_?NumericQ, v_?NumericQ, delta_?NumericQ]
          := odey[x] /.  soln[v, delta][[1]]
```

Taking $v = 0$ and $\delta = 2$, the numerical approximator returns an error with a message

```
NDSolve:  At odex == 1.7173152124676099', step size is effectively
zero; singularity or stiff system suspected.
```

This confirms my suspicion of finite time blow up, and the blow up should be just after $\delta = 1.717$. Careful plotting of the solution and computing of values confirms the expected behavior. In particular, one finds $y(1.717) \doteq 6.03871(10)^7$. I've plotted this solution in Figure 2 along with solutions corresponding to

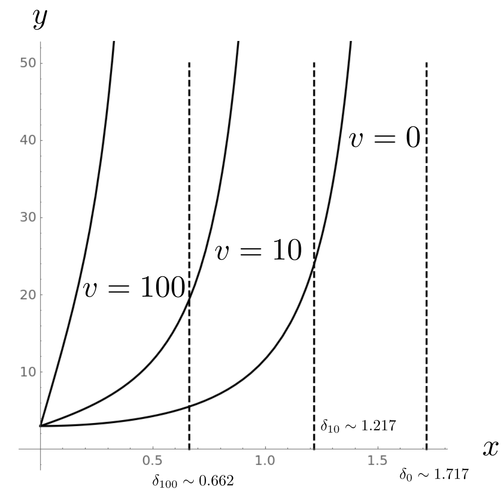

Figure 2: Solutions of $y'' = y^2$, $y(0) = 3$, $y'(0) = v$ illustrating finite time blow-up and the monotone dependece of the blow-up time on the initial (derivative) value.

$y'(0) = v = 10$ and $y'(0) = v = 100$ along with the asymptotes in each case determined approximately from the eror messages from the ODE solver.

Note: The conclusion/guess first discussed above based on comparison with the ODE $y' = y^2$ could have been and may be reached directly using a numerical exploration as suggested in the statement of this part **(b)** of the problem. I intend that this is indicated in Figure 2, though more extensive numerical evidence may be helpful in reaching the basic conclusion/guess.

(c) Multiply both sides of the ODE by y' and integrate to obtain an implicit solution. Does this tell you anything decisive about the assertion of part **(b)**?

The parts of the solution I have typed up above should represent concepts you can relatively easily understand and work you can relatively easily do, though you may not quite have done it or done it in as much detail as I have. This part is a little more challenging, but if you think about it carefully, there is nothing here that is beyond your abilities either. Here I am going to try to justify the guess from part **(b)** which may have been motivated by numerical evidence. Following the hint/instructions given in this part we find

$$y'y'' = y^2 y' \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{1}{2}(y')^2\right) = \frac{d}{dx}\left(\frac{1}{3}\,y^3\right).$$

Integrating from $x = 0$ to x gives

$$\frac{1}{2}(y')^2 - \frac{1}{2}v^2 = \frac{1}{3}y^3 - 9 \qquad \text{or} \qquad (y')^2 = \frac{2}{3}y^3 - 18 + v^2.$$

Notice that since y starts with $y(0) = 3$, the polynomial function

$$\frac{2}{3}y^3 - 18 + v^2$$

of y is always positive for $x > 0$. Consequently, we can take the square root and get the first order ODE

$$y' = \sqrt{\frac{2}{3}}\,\sqrt{y^3 - 27 + 3v^2/2} \qquad \text{or} \qquad \frac{y'}{\sqrt{y^3 - 27 + 3v^2/2}} = \sqrt{\frac{2}{3}}.$$

Integrating again from $x = 0$ to x and changing variables using $\eta = y$ on the left, we find

$$\phi(y) = \int_3^y \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\,d\eta = \sqrt{\frac{2}{3}}\,x. \tag{8}$$

The function $\phi : [3, \infty) \to [0, \infty)$ is increasing because (if we think of y as an independent variable for a moment)

$$\phi'(y) = \frac{d\phi}{dy}(y) = \frac{1}{\sqrt{y^3 - 27 + 3v^2/2}} > 0 \qquad \text{for} \qquad y > 3$$

and satisfies $\phi(3) = 3v^2/2 \geq 0$. This means ϕ has a well-defined inverse $\phi^{-1} : [3v^2/2, w) \to [3, \infty)$ where $w > 3v^2/2$ is some value we don't know right now and, in principle, could also be the extended real number $+\infty$. In any case, knowing the inverse we can write

$$y = \phi^{-1}\left(\sqrt{\frac{2}{3}}\,x\right)$$

to solve the equation on some interval $[0, \delta)$ with $\delta > 0$. We do not know the value of $\delta \in (0, \infty]$ either. This depends on

$$w = \lim_{y \nearrow \infty} \phi(y).$$

To see how this is going to work[2] or at least how it might work, let's go back and consider what we did with the ODE $y' = y^2$ a bit more carefully. That was a first order equation, and we could integrate to obtain

$$-\frac{1}{y} + \frac{1}{y_0} = x. \tag{9}$$

This was because the function y'/y^2 was easy to integrate. We ran across something more complicated and not so easy to integrate explicitly in (8) so we introduced the function ϕ and it's inverse ϕ^{-1} to solve the equation. Let's try that here: We find a function $\phi : [y_0, \infty) \to [0, \infty)$ by

$$\phi(y) = \int_{y_0}^{y} \frac{1}{\eta^2}\, d\eta.$$

The value of this function is on the left in (9). This function is also increasing with $\phi(y_0) = 0$ and

$$\lim_{y \nearrow \infty} \phi(y) = \frac{1}{y_0}.$$

In this case, we know the limit. It is a finite positive value, and it tells us the domain of ϕ^{-1}, namely the interval $[0, 1/y_0)$. The solution

$$y = -\frac{1}{x - 1/y_0}$$

may also be expressed as $y = \phi^{-1}(x)$ and thus, we know the domain of definition of the solution y is $[0, 1/y_0)$ as well with

$$\lim_{x \nearrow 1/y_0} y(x) = +\infty.$$

That is, we have finite time blow-up at $x = 1/y_0$.

Now let's go back to $y'' = y^2$. The function $\phi : [3, \infty) \to [3v^2/2, \infty)$ with values given in (8) is increasing, and we really need to know the value

$$w = \lim_{y \to \infty} \phi(y) = \int_3^\infty \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta$$

[2] or what we can properly do with it

or at least if this positive value is finite. In fact, it is finite. This can be seen for example because there is some $\eta_0 > 3$ for which

$$\frac{1}{2}\eta^3 > 27 - 3v^2/2 \qquad \text{whenever} \qquad \eta > \eta_0.$$

Thus,

$$\begin{aligned} w &= \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta + \int_{\eta_0}^{\infty} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta \\ &< \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta + \int_{\eta_0}^{\infty} \frac{1}{\sqrt{\eta^3/2}}\, d\eta \\ &= \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta + \sqrt{2}\int_{\eta_0}^{\infty} \frac{1}{\eta^{3/2}}\, d\eta \\ &= \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta - 2\sqrt{2}\left(\frac{1}{\sqrt{\eta}}\right)\Big|_{\eta_0}^{\infty} \\ &= \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta + \frac{2\sqrt{2}}{\sqrt{\eta_0}} \\ &< \infty. \end{aligned}$$

We conclude that ϕ increases to a finite limit w with $3v^2/2 < w < \infty$. Consequently,

$$\lim_{\xi \nearrow w} \phi^{-1}(\xi) = +\infty,$$

and the maximal domain of definition (a, b) for a solution of (3) with $y'(0) = v \geq 0$ satisfies

$$b = \delta(v) = \sqrt{\frac{3}{2}}\, w$$

where

$$w = w(v) = \int_3^{\infty} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta < \infty. \tag{10}$$

Thus, these solutions do indeed display finite time blow-up at

$$x = b = \delta(v) = \sqrt{\frac{3}{2}}\, w.$$

It is still not clear that these values b are decreasing with increasing v as suggested by the numerics nor that they satisfy

$$\lim_{v \nearrow \infty} \delta(v) = 0 \tag{11}$$

so that the assertion of part **(b)** is definitively incorrect.

For the monotonicity, we can attempt to differentiate w with respect to v. This yields

$$\frac{dw}{dv} = -\frac{3}{2} \int_3^\infty \frac{v}{(\eta^3 - 27 + 3v^2/2)^{3/2}} \, d\eta < 0$$

so indeed it is the case that w and $\delta(v) = \sqrt{3/2}\, w$ decreases as a function of v.

Finally, to see (11) take any $v_0 > 0$ fixed and consider $v > v_0$. Because the value of the improper integral (10) is finite, given any $\epsilon > 0$ there is some $\eta_0 > 3$ for which

$$\int_{\eta_0}^\infty \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}} \, d\eta < \int_{\eta_0}^\infty \frac{1}{\sqrt{\eta^3 - 27 + 3v_0^2/2}} \, d\eta < \frac{\epsilon}{2}.$$

Now we can take $v_1 > v_0$ so that for all η with $3 < \eta < \eta_0$ and $v > v_1$ we have

$$\frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}} < \frac{\epsilon}{2\eta_0}$$

so that for $v > v_1$

$$\begin{aligned}
w &= \int_3^\infty \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}} \, d\eta \\
&= \int_3^{\eta_0} \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}} \, d\eta + \int_{\eta_0}^\infty \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}} \, d\eta \\
&< \int_3^{\eta_0} \frac{\epsilon}{2\eta_0} \, d\eta + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2\eta_0}(\eta_0 - 3) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2\eta_0}(\eta_0) + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This shows

$$\lim_{v\nearrow\infty} w = 0$$

and establishes (11).

The foregoing solution, especially the solution of part **(c)** may differ substantially from what you submitted as solutions. Your answer to part **(c)** in particular may have been something like: Following the suggestion

$$y'y'' = y^2y',$$

and integrating once I get

$$(y')^2 = v^2 + \frac{2}{3}y^3 - 18.$$

At this point you may have blindly taken the square root to obtain the first order ODE

$$\frac{y'}{\sqrt{y^3 - 27 + 3v^2/2}} = \sqrt{\frac{2}{3}}.$$

And you may have indeed observed that this defines the solution "implicitly" in terms of an integral:

$$\int_3^y \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta = \sqrt{\frac{2}{3}}\; x$$

as in (8) above. This is where things are usually left in an elementary course on ODEs. I guess it is unlikely any of you proceeded to the analysis of the function ϕ defined by

$$\phi(y) = \int_3^y \frac{1}{\sqrt{\eta^3 - 27 + 3v^2/2}}\, d\eta$$

and the remainder of my solution which pretends (at least) to give a definitive answer to the question of part **(b)**. I say "pretends" because I may have made errors in my solution. No one has checked it, and quite honestly it is fairly likely there is one or more errors there. (I have already found and corrected some reading over the solution myself.) On the other hand, I think the basic ideas are there, and if you are interested and willing, I would be very happy for you to check the details and find any errors.

But my main point is that your answer, I suspect, may have ended somewhere with an honest admission that you had little idea how to definitively evaluate the assertion of part **(b)**. For you that may have been the "correct" answer to part **(c)**.

Indeed, it may be that you had no idea of how to test the assertion of part **(b)** either heuristically using the equation $y' = y^2$ and the existence and uniqueness theorem or numerically. That is fine too. You should just express your answer(s) clearly and honestly for yourself. At least that is my suggestion.

Above all, you can think about problems like this and try to understand what they are asking and how to go about answering them.

Problem 5 (regularity) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. Show that if $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$, then f is continuous at $x_0 \in (a, b)$.

Problem 6 (open set) A set $U \subset \mathbb{R}^n$ is said to be an **open set** if for each $\mathbf{p} \in U$ there exists some $r > 0$ so that

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\} \subset U. \tag{12}$$

The set $B_r(\mathbf{p})$ defined in (12) is called an **open ball**.

(a) Show that an open interval (a, b) is an open set in $\mathbb{R}^1$.

(b) Show that an open ball is an open set.

(c) Show that the intersection

$$\bigcap_{j=1}^{k} U_j = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U_j,\ j = 1, 2, \ldots, k\}$$

where $U_1, U_2, \ldots, U_k$ are open sets in $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$, i.e., any intersection of finitely many open sets is an open set.

(d) Show that the intersection of infinitely many open sets need not be an open set.

Problem 7 (uniqueness) Consider the IVP

$$\begin{cases} y' = \sqrt{|y|}, & x \in \mathbb{R} \\ y(0) = 0 \end{cases} \tag{13}$$

and the function $y_1 : \mathbb{R} \to \mathbb{R}$ given by

$$y_1(x) = \begin{cases} (1/4)x^3/|x|, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) What does Theorem 1 tell you about the solution(s) of (13)? In particular:

- **(i)** In order to apply Theorem 1 to the IVP (13), identify an appropriate function $f : \mathbb{R} \to \mathbb{R}$ for which $y' = f$.
- **(ii)** Does f satisfy the hypothesis of the theorem in the case $n = 1$? Why or why not?

(b) Show $y_1 \in C^1(\mathbb{R})$. In particular:

- **(i)** Draw the graphs of y_1 and y_1'; properly label the axes.
- **(ii)** Draw the graph of f from part **(a)** above; properly label the axes.

(c) Show y_1 satisfies (13).

(d) Find three other solutions of (13) and draw the graphs of two of the solutions you find.

Problem 8 (another IVP) Consider the IVP

$$\begin{cases} y' = y^2, \\ y(t_0) = y_0. \end{cases} \tag{14}$$

(a) What does Theorem 2 tell you about the solution(s) of (14)?

(b) Solve (14). Hint: The ODE is separable.

(c) For each $(y_0, t_0) \in \mathbb{R}^2$ there exists a unique smallest extended real number $a \in [-\infty, t_0)$ and a unique largest real number $b \in (t_0, \infty]$ for which (14) has a unique solution on the interval (a, b). Find a and b (as functions of t_0 and y_0).

Problem 9 (linear IVP) Consider the IVP

$$\begin{cases} y'' = y, \\ y(1) = 1. \end{cases} \tag{15}$$

(a) What does Theorem 2 tell you about the solution(s) of (15)? In particular:

(i) In order to apply Theorem 2 to the IVP (15), identify an appropriate matrix valued function $A : \mathbb{R} \to \mathbb{R}^{2\times 2}$ for which the ODE in (15) is equivalent to $\mathbf{x}' = A\mathbf{x}$.

(ii) Identify the initial point $\mathbf{p}$ for application of Theorem 2.

(b) Solve the IVP for the system you identified in part **(a)** above.

(c) Solve (15).

(d) Plot at least three different solutions of (15). You may wish to use mathematical software like Matlab, Maple, or Mathematica. Why does this not violate the uniqueness assertion of Theorems 1 and 2?

Problem 10 (two point boundary value problem) Given $L > 0$, a function $f \in C^0[0, L]$, and $c, d \in \mathbb{R}$ consider the BVP

$$\begin{cases} y'' = f(x), & x \in (0, L) \\ y(0) = c, \\ y(L) = d. \end{cases} \tag{16}$$

Find a function $g \in C^0[0, L]$ so that the BVP (16) is **equivalent** to the BVP

$$\begin{cases} u'' = g(x), & x \in (0, L) \\ u(0) = 0, \\ u(L) = 0. \end{cases} \tag{17}$$

Once you find g, complete the following:

(a) Given the unique solution u of (17) you can find a formula for the unique solution y of (16) in terms of u.

(b) Given the unique solution y of (16) you can find a formula for the unique solution u of (17) in terms of y.

Hint: The function g should depend on f, c and d.